TWO WORKS ON ITERATION AND RELATED QUESTIONS

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I want to give a brief account of results which have been communicated to me and which have been obtained by two young geometers, Efi Jabotinsky and Michel Lüntz. Mr. Lüntz is unhappily in a concentration camp in France. I must add that this is one of the reasons for the fact, for which I must apologize, that I am writing this exposé only now, although I have had both works in my possession for several months. The impossibility of communicating with Mr. Lüntz and therefore of asking him for any explanation made the examination of his paper especially difficult.

As is well known, the problem of iteration, a function $f(x)$ being given, consists in finding a one-parameter family of functions $f_n(x) = f(n,x)$ such that, for $n=1$, we have

$$f(1, x) = f(x) \tag{1}$$

and, moreover, for any m, n,

$$f_m[f_n(x)] = f_{m+n}(x); \tag{2}$$

also

$$f(0, x) = x \tag{3}$$

which follows from (2) as is seen by taking $m=0$.

The question is connected with Abel's functional equation

$$\phi[f(x)] = 1 + \phi(x) \tag{A}$$

because if ϕ is a solution of that equation a solution of (1), (2) will be given by

$$\phi[f_n(x)] = n + \phi(x). \tag{4}$$

Instead of (A), one can introduce Schroeder's equation

$$\psi[f(x)] = k\psi(x) \tag{A'}$$

(k a constant) in which ψ is connected with the unknown ϕ of (A) by $\psi = k^{\phi}$, and with the help of which the solution would be expressed by

$$\psi[(f_n(x)] = k^n\psi(x).$$

The fact that every solution of (A) gives a solution of the iteration

Presented to the Society, April 24, 1943, under the title *On fractional iteration and connected questions*; received by the editors July 1, 1943.

problem is sufficient to show that the latter is indeterminate unless proper conditions are added: for, if $\phi(x)$ is a solution of (A), so is also $\phi(x)+\Pi[\phi(x)]$ where Π is any function admitting of the period 1.

The iteration problem has been considered by several authors. The most recent work, as far as I know, is due to Paul Lévy (C. R. Acad. Sci. Paris vol. 184 p. 500 and Annali di Matematica (4) vol. 5 (1928) pp. 270–296, esp. p. 287 ff.). Additional conditions being necessary (as a matter of fact, conditions of regularity), Paul Lévy imposes such conditions either (1) at infinity—a difficult subject, which we shall leave aside—or (2) in the neighborhood of a determinate value, which the successive iteratives of f are assumed to approach. In the latter case, the one which will interest us, he uses the classic studies of Koenigs (Ann. École Norm. 1884).

One feature of the problem is not considered by the above mentioned authors, namely that, on account of (2), the set of the various transformations $x \to f(n, x)$ is a one-parameter group, which can be defined by its infinitesimal transformation $\delta x = \xi(x)\delta t$. The question has been faced from this point of view in a note of Bouton[1] for the case in which the transformations are assumed to be tangent to the identical one. Moreover, that group aspect and the infinitesimal transformation have also been introduced in the fundamental memoir of G. D. Birkhoff,[2] though without playing an essential role.

Jabotinsky begins by considering not one problem of iteration, but the general class of them, that is, the equations (2) and (3) without immediate consideration of any special function $f(x)$ nor, therefore, of equation (1). Now all these problems satisfy a common partial differential equation. For, writing (2) in the form

$$f[(m-n), f(n, x)] = f(m, x)$$

and expressing the fact that the left-hand member does not depend on n, Jabotinsky easily obtains the partial differential equation

$$(\mathrm{E}) \qquad (\partial^2 f/\partial n^2)(\partial f/\partial x) - (\partial^2 f/\partial n \partial x)(\partial f/\partial n) = 0.$$

Conversely, this equation admits of the two intermediate integrals

$$(5) \qquad (\partial f/\partial n):(\partial f/\partial x) = \xi(x),$$

$$(6) \qquad \partial f/\partial n = \eta(f)$$

[1] Bull. Amer. Math. Soc. vol. 23 (1917) p. 75: A scanty note, the development of which has not been published as far as we know.

[2] Acta Math. vol. 43 (1922) p. 1. As is well known, that memoir deals with transformations in two variables, the existence of an integral invariant being assumed. See also the paper of Daniel C. Lewis, Duke Math. J. vol. 5 (1939) p. 794 in connection with G. D. Birkhoff's ideas.

(ξ, η, arbitrary functions) so that its general solution is

$$(7) \qquad f(n, x) = \chi[n + \phi(x)] \qquad \left(\phi(x) = \int dx/\xi(x), \chi \text{ arbitrary}\right).$$

If we add condition (3), we see that χ must be the inverse function of ϕ, so that the above formula brings us back to (A). Thus we have a simple proof of the fact that not only every solution of (A) gives us a solution of (2), (3), but that, conversely,[3] every solution of (2) corresponds to a solution of (A).

The partial differential equation (E) is common to every problem of the kind which interests us, whatever the given function $f(x)$ may be.

In order to integrate that equation, (3) gives the condition $f(0, x) = x$; and, if we now introduce (1), we see that we are also given the values of $f(n, x)$ for $n = 1$. It is a boundary problem of an unusual form; and, indeed, we know that our problem is not determinate if we do not add complementary conditions of regularity.

As was already done in the aforesaid works of Koenigs and Paul Lévy, Jabotinsky investigates what will happen in the neighborhood of a "double point," that is of a value of x—say $x = 0$—such that

$$(8) \qquad f(x) = x,$$

and he limits himself to iterates $f(n, x)$ of $f(x)$ such that every $f(n, 0)$ is also equal to 0, a hypothesis which, as we shall see, hardly diminishes generality. Moreover, he assumes f to be expansible in powers of x and tries to find a corresponding expansion for f_n, say

$$(9) \qquad f_n(x) = A_1(n)x + A_2(n)x^2 + \cdots.$$

This is what Paul Lévy has carried out in his note in the C. R. Acad. Sci. Paris for the case of $f(x) = e^x - 1$ (of which Jabotinsky was not informed). For that purpose, Jabotinsky substitutes (9) into (E) or, preferably, into the intermediate integral (5). If $\xi(x) = \alpha_1 x + \alpha_2 x^2 + \cdots$ were known, this would give, for the $A_i(n)$, the successive linear differential equations of the first order

$$(10) \qquad \begin{aligned} A_1'(n) &= \alpha_1 A_1(n), \\ A_2'(n) &= 2\alpha_1 A_2(n) + \alpha_2 A_1(n), \\ A_3'(n) &= 3\alpha_1 A_3(n) + 2\alpha_2 A_2(n) + \alpha_3 A_1(n), \\ &\cdots\cdots\cdots\cdots\cdots\cdots \end{aligned}$$

[3] Around a double point (which case we consider below), it is necessary to replace (A) by Schroeder's equation (A'), as does Koenig in the above cited memoir.

to be integrated with the initial conditions $A_1(0)=1$, $A_i(0)=0$ for $i \geqq 2$—an elementary integration.

Now, on account of (3), $\xi(x)$ is nothing else than the derivative of $f(n, x)$ with respect to n for $n=0$, so that if ξ were known, this would come back to solving Cauchy's problem for (E). From the point of view of group theory, $\xi(x)$ *corresponds to the infinitesimal transformation of the group which interests us.*[4]

As a matter of fact, in our case the α's are auxiliary unknowns; they are to be determined by the knowledge of the values of the A's for $n=1$, which proceeds without any difficulty, for each i, after the integration of the successive equations (10). If $\alpha_1 \neq 0$, that is if $a_1=A_1(1) \neq 1$, the expressions thus obtained obviously contain n through polynomials in powers of $e^{\alpha_1 n}=a_1{}^n$, so that they are quite similar to those which G. D. Birkhoff has constructed in the case which he has investigated.

The case $a_1=1$ is comparable to Birkhoff's case II'', the successive unknowns A in (10) being polynomials in n.

But, as in Birkhoff's case, there remains the question whether the A_i' thus calculated would give a convergent series for (9); and, precisely, Birkhoff has shown, by a remarkable example (see §31, p. 55 of his memoir), that the contrary case may occur in the problem which he has treated. Whether the same fact is possible for the transformations in one variable remains to be decided.

Lüntz's starting point is slightly different: he investigates "mutually reversible" functions, two functions f and g being said to be reversible to each other[5] if

$$f[g(x)] = g[f(x)]. \tag{11}$$

He points out that this is a kind of generalization of fractional iteration; and, indeed, we see at once by (2) that any two of the f_m are reversible to each other. Conversely, will the investigation of func-

[4] If the transformation $x \to f(x)$ were known to belong to a certain group of Lie, G_r, in a finite number r of parameters, our question would be considered as being treated by Elie Cartan in his memoir on *Geométrie des groupes de transformations* (Annali di Matematica (9) vol. 6 (1927) p. 1). In Cartan's terminology, a one-parameter group in G_r is the analogue of a geodesic in the G_r-space, which would be defined by its origin (the identical transformation) and its initial direction (defined by the corresponding infinitesimal transformation); our problem would correspond to drawing a geodesic through two given points. The solution is elementary in Cartan's case, but of a quite different difficulty when no G_r containing the given transformation is known.

[5] Lüntz is right in not using the word "permutable," which is used by Volterra in the *Functional calculus* with a different meaning. On the contrary, the transformations $x \to f(x)$ and $x \to g(x)$ are said to be permutable.

tions g reversible to a given function f always lead to the iterates of f?

We shall presently see that, under the above postulated regularity conditions, such will be the case. But it must be noticed that this new problem is, from a certain point of view, simpler than the former since, instead of a family of functions, we investigate a single function g.

As was done above, Lüntz also starts from a double point x_0 of f, say $x_0=0$. Now, taking $x=x_0=0$ in the fundamental relation (11), we see that $g(x_0)$ must be a double point of f also, so that Lüntz is led to admit that $g(x_0)=g(0)$ is nothing else than x_0 itself.

Lüntz also remarks that if g_1 and g_2 are both reversible to f, so is also $g_1[g_2(x)]$, so that (a well known fact in the transformation theory) the g's are the elements of a group. He, moreover, supposes that such functions g_1 and g_2 are also reversible to each other, which we shall see to be the case.[6]

Again, taking $x_0=0$ and representing f and g by expansions in powers of x,

$$f(x) = a_1x + a_2x^2 + \cdots, \tag{12}$$

$$g(x) = b_1x + b_2x^2 + \cdots, \tag{13}$$

(11) gives the relations

$$\begin{aligned} &a_1b_1 = b_1a_1, \qquad a_1b_2 + a_2b_1^2 = b_1a_2 + b_2a_1^2, \\ &a_1b_3 + 2a_2b_1b_2 + a_3b_1^3 = b_1a_3 + 2b_2a_1a_2 + b_1a_1^3, \\ &a_1b_4 + a_2(2b_1b_3 + b_2^2) + 3a_3b_1^2b_2 + a_4b_1^4 \\ &\qquad = b_1a_4 + b_2(2a_1a_2 + a_2^2) + 3b_2a_1^2a_2 + b_4a_1^4 \\ &\quad \cdots\cdots\cdots\

Moreover, the above expression of the unknown is not written in terms of the a's but, as previously, implies an auxiliary set of parameters β, the meaning of which would not be obvious by itself, but appears by comparison with Jabotinsky's result. Indeed, giving the terms of the first degree in $\lambda-1$, they are, but for numerical factors, nothing else than the α in (10), so that the above formula leads us back to Jabotinsky's calculation.

Again, these parameters are to be found in terms of the a's, and the convergence question is left aside.

If we assume a_1 to be equal to 1 and if $a_2 \neq 0$, the coefficient b_1 will no longer be arbitrary: it must be equal either to 0 or to 1, so that the expansion of g will be ($b_1=0$ being again excluded) of the form $x+\cdots$, and will again include an arbitrary parameter λ, namely the coefficient b_2 of x^2.

The third, fourth, $\cdots$ equations (14) allow us to calculate the successive b's. Again Lüntz is led to introduce a set of parameters which coincide with the coefficients in the infinitesimal transformation of the group and, therefore, to write the same formula as would be found by Jabotinsky's method.

Consequently, we see that for $a_1 \neq 0, 1$, as for $a_2 \neq 0$, the family of functions reversible to f and satisfying our regularity conditions is not distinct from the family of iterates of f.

These are the results obtained by Jabotinsky and Lüntz; they coincide with each other, at least if we do not have $a_1=0$ or $a_1=1$, $a_2=0$, though they are deduced from rather different points of view.

This coincidence suggests several remarks, some of which I shall point out now.

I. In what precedes, we have started from the hypothesis that a double point $x=x_0$ of $f(x)$ is common to all iterates $f(n, x)$ or to all the functions $g(x)$ reversible to f. As we have seen, Lüntz makes the important remark that, on account of the definition (11) of reversible functions, the transform $g(x_0)$ is also a double point of f. He leaves, however, one question open, namely whether this transform is x_0 itself or another solution of (8). But, as a matter of fact, this question can be answered easily if the function g is assumed to belong to a continuous family of functions all of which are reversible to f, one of them being $g(x)=x$; for, substituting x_0 in each of these functions, we obtain results which, varying continuously, can not be different from x_0 itself, at least if the latter is an isolated zero of $f(x)-x$.

This conclusion immediately applies to the iteration problem, as any solution of it is a solution of Lüntz's problem, so that at least

some information on this subject is obtained by Lüntz's way of attacking it.

II. Is the above assumption—namely that either $a_1 \neq 0$, 1 or $a_1 = 1$, $a_2 \neq 0$—essential for the general behaviour of the results and especially for the equivalence of those problems?

This would seem to be presumable a priori: for it is evident that if the function $f(x)$ would reduce to x itself, Lüntz's problem would become fully indeterminate, as the function $f(x) = x$ is reversible to every other one.

A most remarkable fact is that this presumption is not verified. Let the first term in (12) be reduced to x, the first following term which does not vanish being of degree p, so that

$$f(x) = x + a_p x^p + \cdots \qquad (p \geqq 2,\ a_p \neq 0).$$

Equating the coefficients of x^p in both members of (11), we immediately see that $b_1 = b_1^p$. Let us take, therefore,[7] $b_1 = 1$, so that the expansion of $g(x)$ is also of the form

$$g(x) = x + b_r x^r = \cdots \qquad (r \geqq 2,\ b_r \neq 0).$$

If so, all terms issuing from the first term of (12) or from the first term of (13) annul each other. After them, the terms of lowest degree are in x^{p+r-1}, with coefficients respectively equal to $pa_p b_r$ and $ra_p b_r$ in both members, so that r can not be different from p. Then, b's with suffixes greater than p would be determined by equations some of which (except for $p = 2$) will reduce to the form $a_p b_{p+q} = b_p a_{p+q}$ while, for $p = 2$, we shall have

$$a_2 b_3 - a_3 b_2 = a_2 b_2 (b_2 - a_2).$$

We see that, in every case, the functions reversible to $f(x)$ and regular at $x = 0$ constitute a one-parameter family, which necessarily coincides with the iterates of f.

We also see that Lüntz was right in supposing that two functions reversible to the same third one are reversible to each other. This, as is classic in the group theory, carries the consequence that the reversibility conditions can be expressed by the existence of *invariants*. Let c_1 be chosen once for all, different from 0 and 1: we can consider as a "reduced" form of $g(x)$ the one which has c_1 as coefficient of x. Every other coefficient in that reduced expansion will be deter-

[7] Provisionally, we still exclude, like Lüntz, the value zero and, to simplify discussion, do not consider roots of unity. At any rate, in iteration problems it is natural (cf. Birkhoff, loc. cit. p. 9) to eliminate negative first coefficients.

mined by these conditions and, on the other hand, we just saw that the reduced g will be reversible not only to f, but also to any other function (13) reversible to f, so that such successive coefficients of the reduced expansion, when calculated in terms of the a's, must have the same values as the ones similarly calculated in terms of the b's: they give invariants (which, obviously, only seemingly depend on the special value of c_1). The first of them are

$$j_2 = a_2 : (a_1 - a_1^2), \qquad j_3 = (a_3 - 2a_2 j_2) : (a_1 - a_1^3),$$
$$j_4 = \{a_4 - [(3a_1 + 5)j_3 + 5j_2^2]a_2\} : (a_1 - a_1^4),$$
$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots.$$

Such invariants are evidently related to the coefficients of the infinitesimal transformation of the group—a connection which, however, we shall not investigate.

In the case, considered in II, of $a_1 = 1$, the first invariant, for $p = 2$, is

$$a_3/a_2 - a_2 \tag{15}$$

while, for $p \geqq 2$, one or several invariants will be of the form $a_{p+q} : a_p$, after which there will come one analogous to (15).

III. The transformations $S : x \to f(x)$, $T : x \to g(x)$, where f and g are connected by Lüntz's relation (11), being permutable to each other, the transform of T by S coincides with T. In other words, the curve $y = g(x)$ is invariant by the point transformation $X = f(x)$, $Y = f(y)$. The general problem of invariant points has been studied several times, especially by Poincaré, Samuel Lattés,[8] G. D. Birkhoff (loc. cit.) and the present author; but the present transformation behaves in an abnormal manner from this point of view, as there exist an infinite number of invariant curves, namely every nth iterate $y = f_n(x)$, such iterates existing at least for every integral n.

IV. The problem of reversible functions being connected with the iteration problem, we need not wonder at seeing it connected with Abel's equation (A). As a matter of fact, this connection is a very simple one. Let ϕ be any solution of (A) and let us apply the operation ϕ to both members of (11). We immediately see that the function

$$\Phi(x) = \phi[g(x)] \tag{16}$$

is again a solution of (A). Conversely, if ϕ and Φ be two solutions of

[8] Poincaré, J. Math. Pures Appl. (4) vol. 1 (1885); Hadamard, Bull. Soc. Math. France vol. 29 (1901); Samuel Lattés, Annali di Matematica (3) vol. 13 (1907).

(A), the function g defined by (16)—in other words, $g(x)=\phi^{-1}[\Phi(x)]$—is reversible to f. Any determinate function $g(x)$ permutable with $f(x)$ can be obtained in an infinite number of ways by this procedure.

V. We have noticed—see footnote (4)—that if we knew a finite Lie group G containing the transformation $x\to f(x)$, we could deduce therefrom the one-parameter group (9) containing that same transformation. Now, in G. D. Birkhoff's example, the one-parameter group analogous to (9) does not exist (otherwise than formally). This would suggest that the corresponding transformation is contained in no finite Lie group. However, there is the objection that the deduction of (9) from G, by Elie Cartan's method, might be valid only in a local domain, Birkhoff's transformation lying beyond that domain.

VI. The case $a_1=0$, which Lüntz has explicitly excluded, is also implicitly excluded from Jabotinsky's study. Indeed, the impossibility of any regular solution of the form (9) at once appears from the first equation (10); as for any solution of that equation, the conditions $A_1(0)=1$ and $A_1(1)=0$ are contradictory. Therefore, no function which is regular around $x=0$ and whose expansion begins by a term of degree 2 or higher can belong to a regular continuous one-parameter group of Lie; nor can it, as we saw, belong to a regular finite group in any number of parameters.

Now, such a function f of that kind being given, let us consider the reversibility problem. In the expansion (13) of a function g reversible to f, we shall have, as previously, either

(1) $b_1=1$: one easily sees that this gives nothing else than $g(x)$ identically equal to x; or

(2) $b_1=0$: corresponding solutions are evidently $g(x)$ equal to $f(x)$ itself or to one of its iterates.

But there may be solutions other than these trivial ones, for the two functions $f(x)=x^m$ and $g(x)=x^n$ are obviously reversible. From those, or more generally from $f(x)=ax^m$ and $g(x)=bx^n$ (which are reversible if $ab^m=ba^n$), we obviously could deduce other ones seemingly, but not essentially, distinct from them by applying to $x, f(x)$ and $g(x)$ a common point transformation, regular and one-one around $x=0$. Whether this would afford the most general solution relating to our present case, $a_1=0$, remains to be investigated.

Another curious question would arise if equation (8) were to admit two different roots x_0, x_1 around which expansions were to behave differently from the expansions considered above—for example $f'(x_0)=0$, $f'(x_1)=0$—giving rise possibly to special kinds of singularities.

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ON THE EXTENSION OF DIFFERENTIABLE FUNCTIONS

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The author has shown previously how to extend the definition of a function of class C^m defined in a closed set A so it will be of class C^m throughout space (see [1]).[1] Here we shall prove a uniformity property: If the function and its derivatives are sufficiently small in A, then they may be made small throughout space. Besides being bounded, we assume that A has the following property:

(P) There is a number ω such that any two points x and y of A are joined by an arc in A of length less than or equal to ωr_{xy} (r_{xy} being the distance between x and y).

This property was made use of in [2]; its necessity in the theorem is shown by two examples below.

A second theorem removes the boundedness condition in the first theorem, and weakens the hypothesis (P); its proof makes use of the proof of the first theorem. We remark that in each theorem, as in [1], the extended function is a linear functional of its values in A.

The proof of Theorem I is obtained by examining the proof in [1]; hence we assume that the reader has this paper before him, and we shall follow its notations closely.

THEOREM I. *Let A be a bounded closed set in n-space E with the property* (P), *and let m be a positive integer. Then there is a number α with the following property. Let $f(x)$ be any function of class C^m in A, with derivatives $f_k(x)$ ($\sigma_k = k_1 + \cdots + k_n \leqq m$). Suppose*

$$|f_k(x)| < \eta \qquad (x \in A,\ \sigma_k \leqq m).$$

Then $f(x)$ may be extended throughout E so that

$$|f_k(x)| < \alpha\eta \qquad (x \in E,\ \sigma_k \leqq m).$$

Let d be the diameter of A, or 1 if this is larger, and let R be a spherical region of radius $2d$ with its center at a point of A. Set $f(x) = 0$ in $E - R$. Then the extension of f in $R - A$ given in [1] will be shown to have the property, using

$$\alpha = 2n(m!)^n(m+1)^{3n}(433n^{1/2}d\omega)^m cN,$$

where N and c are as given in [1, §§11, 12]. Note that $433 = 4 \cdot 108 + 1$.

Presented to the Society, September 13, 1943; received by the editors November 27, 1943.

[1] Numbers in brackets refer to the references cited at the end of this paper.

Set $B=A\cup(\overline{R}-R)$. We show first that for any points x', x'' of B,

$$|R_k(x';x'')|<\beta r_{x'x''}^{m-\sigma_k}\eta,\qquad \beta=2n(m+1)^n\omega^m.$$

Suppose first that x' and x'' are in A. Let C be a curve in A joining them, of length less than or equal to $\omega r_{x'x''}$. The inequality is then a consequence of [2, Lemma 3]. Suppose next that one of the points is in A, and the other is in $\overline{R}-R$ (the case that both are in $\overline{R}-R$ is trivial). By [1, (3.1)], since $r_{x'x''}\geqq d\geqq 1$,

$$|R_k(x';x'')|\leqq\eta+\sum_{\sigma_1\leqq m-\sigma_k}\eta r_{x'x''}^{\sigma_1}\leqq r_{x'x''}^{m-\sigma_k}\eta\left[1+\sum_{\sigma_1\leqq m-\sigma_k}1\right]$$

$$\leqq(m+1)^n r_{x'x''}^{m-\sigma_k}\eta.$$

Now take any x in $R-B$. Let $\delta^*/4$ be the distance from x to B, and let x^* be a point of B distant $\delta^*/4$ from x. Say x is in the cube C of the set of cubes K_ν; let $I_{\lambda_1},\cdots,I_{\lambda_t}$ be those I_λ with points in C (see [1, §11]). Now y^ν is the center of I_ν, and x^ν is a nearest point of B to y^ν. As noted in [1, (9.1)], $r_{y^\nu x^\nu}$ and $r_{y^\nu x^*}$ each lie between $\delta^*/8$ and $\delta^*/2$. Since $r_{xy^\nu}<\delta^*/2$, we have

$$r_{x^\nu x^*}<\delta^*,\qquad r_{xx^\nu}<\delta^*.$$

The definition of ζ in [1, §11] together with [1, (6.3)] gives

$$\zeta_{\nu;k}(x)=\psi_k(x;x^\nu)-\psi_k(x;x^*)=\sum_{\sigma_1\leqq m-\sigma_k}\frac{R_{k+1}(x^\nu;x^*)}{l!}(x-x^\nu)^l.$$

Hence

$$|\zeta_{\nu;k}(x)|<(m+1)^n\beta r_{x^\nu x^*}^{m-\sigma_k-\sigma_1}r_{xx^\nu}^{\sigma_1}\eta<(m+1)^n\beta\delta^{*m-\sigma_k}\eta.$$

Following [1, §11] still, we find

$$|D_kg(x)-\psi_k(x;x^*)|<c\sum_{\sigma_1\leqq m-\sigma_k}(m!)^n 2^{m\sigma_1}N(m+1)^n\beta\delta^{*m-\sigma_k+\sigma_1}\eta.$$

As in [1], $2^\nu<108n^{1/2}/\delta^*$; hence

$$|D_kg(x)-\psi_k(x;x^*)|<c(m!)^nN(m+1)^{2n}(108n^{1/2})^m\beta\delta^{*m-\sigma_k}\eta.$$

Moreover, since $r_{xx^\nu}<3d$, [1, (6.1)] gives

$$|\psi_k(x;x^*)|<3^m(m+1)^n d^m\eta.$$

Since $\delta^*\leqq 4d$ and $f(x)=g(x)$ in $R-B$, the theorem follows.

We turn now to the second theorem. We shall say A *satisfies* (P) *locally* if for each $x\in A$ there is a neighborhood U of x and a number

ω such that any two points y and z of $A \cap U$ are joined by an arc in A of length not greater than ωr_{zy}.

THEOREM II. *Let A be a closed subset of an open set R in E, satisfying (P) locally, and let m be a positive integer. Then for any continuous function $\epsilon(x)$ defined and greater than 0 in R there is a continuous function $\delta(x)$ defined and greater than 0 in A with the following property. Let $f(x)$ be any function of class C^m in A, such that*

$$|f_k(x)| < \delta(x) \qquad (x \in A, \sigma_k \leqq m).$$

Then $f(x)$ may be extended throughout R so that

$$|f_k(x)| < \epsilon(x) \qquad (x \in R, \sigma_k \leqq m).$$

REMARKS. The preceding theorem is easily seen to be a consequence of this one. The present theorem holds if E is replaced by a differentiable manifold M, in which a fixed set of coordinate systems (each one intersecting but a finite number of others) is used to measure the size of derivatives. To show this, we imbed M in a Euclidean space E' (see [3, Theorem 1]), giving $A \subset R \subset R' \subset E'$ (R' open in E'; we let R' contain no points of the limit set of M), extend f throughout R' (see the proof of [3, Lemma 4]), and consider its values in R.

To prove the theorem, we begin by choosing spherical regions $U_1, U_2, \cdots$, each $\overline{U}_i \subset R$, with the following properties:

(a) Each U_i is in a neighborhood U as described above.

(b) Each $\overline{U}_i$ intersects but a finite number of other $\overline{U}_j$.

(c) If U_i is of radius ρ_i, and U_i' is the concentric region of radius $\rho_i/2$, then $R = \sum U_i'$.

Let $\psi^i(x)$ be a function of class C^m in E such that

$$\psi^i(x) > 0 \qquad (x \in U_i'),$$

$$\psi^i(x) = 0 \qquad (x \in E - U_i').$$

Set

$$\phi^i(x) = \psi^i(x) / \sum \psi^j(x) \qquad (x \in R);$$

then $\phi^i(x)$ is of class C^m in R, and

$$\phi^i(x) = 0 \qquad (x \in R - U_i'),$$

$$\sum \phi^i(x) = 1 \qquad (x \in R).$$

The extension of $f(x)$ is defined as follows. Set

$$f^i(x) = \phi^i(x) f(x) \qquad (x \in A),$$

$$f^i(x) = 0 \qquad (x \in R - U_i).$$

Then f^i is of class C^m in $A \cup (R - U_i)$. Extend it as in [1] (using a fixed

subdivision of $U_i - A$; we could set $f^i(x) = 0$ in $E - R$) to be of class C^m in R (or E). (Note that if $A \cap U_i' = 0$, then $f^i(x) = 0$, $x \in R$.) Set

$$f(x) = \sum f^i(x) \qquad (x \in R).$$

Then f is an extension of class C^m of its values in A. We must show that it satisfies the condition of smallness.

Choose $a_i \geqq 1$ so that

$$|\phi_k^i(x)| \leqq a_i \qquad (x \in R, \sigma_k \leqq m),$$

then if $|f_k(x)| < \eta$ $(x \in A \cap U_i')$,

$$|f_k^i(x)| = \left| \sum_{\sigma_l \leqq \sigma_k} \phi_l^i(x) f_{k-l}(x) \right| \leqq (m+1)^m a_i \eta \qquad (x \in A).$$

By the choice of U_i, there is an ω_i such that any x' and x'' in $A \cap U_i$ are joined by an arc in A of length not greater than $\omega_i r_{x'x''}$. Set $\sigma_i = \max(1, 2/\rho_i)$. If R_k^i is the remainder for f_k^i, we shall show that for any x' and x'' in $A \cup (R - U_i)$,

$$|R_k^i(x'; x'')| < 2n(m+1)^{2n} \omega_i^m a_i \sigma_i^m r_{x'x''}^{m-\sigma_k} \eta.$$

If x' and x'' are both in U_i, we apply [2, Lemma 3]. If $x' \in R - U_i$ and $x'' \in U_i'$, or vice versa, then $r_{x'x''} \geqq \rho_i/2$, and the proof in the preceding theorem applies; we consider separately the cases $\rho_i/2 \geqq 1$, $\rho_i/2 < 1$, using $r_{x'x''} \geqq 1$ and $\sigma_i r_{x'x''} \geqq 1$ respectively. If $x' \in R - U_i$ and $x'' \in R - U_i'$, or vice versa, $R_k^i = 0$, since $\phi_l^i(x') = \phi_l^i(x'') = 0$. The proof of the preceding theorem now shows that for some α_i, if

$$|f_k(x')| < \eta \qquad (x' \in A \cap U_i', \sigma_k \leqq m),$$

then

$$|f_k^i(x)| < \alpha_i \eta \qquad (x \in R, \sigma_k \leqq m).$$

(We may set $\alpha_i = 1$ if $A \cap U_i' = 0$.)

Given $\epsilon(x)$, we determine $\delta(x)$ as follows. For each $x \in R$ there is a set of numbers $\lambda_1, \cdots, \lambda_s$, $s = s(x)$, such that

$$x \in \text{each } U_{\lambda_j}', \qquad x \in \text{no other } U_i'.$$

Because of (b), s is finite. Set $\alpha(x) = \alpha_{\lambda_1} + \cdots + \alpha_{\lambda_s}$.
We can clearly choose a continuous function $\beta(x)$ in R such that

$$\alpha(x) < \beta(x) \qquad (x \in R).$$

We may now choose a continuous function $\delta(x') > 0$ in A such that

for any $x' \in A$, if $U'_{\mu_1}, \cdots, U'_{\mu_s}$ are the U'_j containing x', then

$$\delta(x') \leqq \epsilon(x)/\beta(x) \quad (x \in U'_{\mu_1} \cup \cdots \cup U'_{\mu_s}).$$

Now take any f of class C^m in A, with $|f_k(x)| < \delta(x)$ $(x \in A, \sigma_k \leqq m)$; the extension of f through R has been defined. Take any $x \in R$; define $\lambda_1, \cdots, \lambda_s$ as above. Then

$$|f_k(x')| < \delta(x') \leqq \epsilon(x)/\beta(x) \qquad (x' \in A \cap U'_{\lambda_j}, \sigma_k \leqq m),$$

and hence

$$|f_k^{\lambda_j}(x)| < \alpha_{\lambda_j}\epsilon(x)/\beta(x) \qquad (\sigma_k \leqq m).$$

Since $f_k(x) = f_k^{\lambda_1}(x) + \cdots + f_k^{\lambda_s}(x)$,

$$|f_k(x)| < \alpha(x)\epsilon(x)/\beta(x) < \epsilon(x)$$

for $\sigma_k \leqq m$, which completes the proof.

EXAMPLES. (1) Let A consist of a point, together with a sequence of points approaching it. Letting $f(x) = 1$ at a finite number of points of the sequence, and $f(x) = 0$ in the rest of A shows (with $m = 1$) that the theorem fails here.

(2) Let A be the closed region of the plane defined by (a) $x^2 + y^2 \leqq 1$, and (b) either $x \leqq 0$ or $|y| \geqq x^{3/2}$. Let $f(x, y) = 0$ if $x \leqq 0$, and set

$$f(x, y) = \begin{cases} \gamma x^2/(1 + \gamma^2 x^2) & \text{if } x \geqq 0,\ y > 0, \\ -\gamma x^2/(1 + \gamma^2 x^2) & \text{if } x \geqq 0,\ y < 0. \end{cases}$$

We see easily that f is of class C^1 in A. (It would not be if, in (b), we used $|y| \geqq x^2$.) The maximum $\partial f/\partial x$ occurs at $x = 1/(3^{1/2}\gamma)$, and has the value $9/(8 \cdot 3^{1/2})$. Set

$$p = (1/3^{1/2}\gamma,\ 1/3^{3/4}\gamma^{3/2}), \qquad q = (1/3^{1/2}\gamma,\ -1/3^{3/4}\gamma^{3/2}).$$

Then

$$\frac{f(p) - f(q)}{r_{pq}} = \frac{2\gamma/3\gamma^2}{1 + \gamma^2/3\gamma^2} \div \frac{2}{3^{3/4}\gamma^{3/2}} = \frac{3^{3/4}}{4}\gamma^{1/2}.$$

Hence, in any extension of f through the plane, we must have $|\partial f/\partial y| \geqq 3^{3/4}\gamma^{1/2}/4$ at some point (between p and q); yet $|f|$, $|\partial f/\partial x|$ and $|\partial f/\partial y|$ are uniformly bounded for all $\gamma > 1$. Taking γ arbitrarily large shows that the conclusion of the theorem does not hold.

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THE SYMMETRIC JOIN OF A COMPLEX

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1. **The definition of J.** Let K be a polyhedron. With each pair of distinct points p, q of K we associate a closed line segment pq. No distinction is made between p and q and the corresponding end points of pq. The length of pq is a continuous function of p and q, and the length approaches zero if p and q approach a common limit. Distinct segments do not intersect except at a common end point. The points of these segments with their obvious natural topology make up J, the symmetric join of K. This space arises in [4][1] in connection with the problem of finding the chords of a manifold that are orthogonal to the manifold.

2. **The subdivision of J.** Let the mid-point of pq be denoted by $\Lambda p\times q=\Lambda q\times p$, and let $p=\Lambda p\times p$. These points $\Lambda p\times q$ make up the symmetric product S of K. Let the mid-point of the segment from p to $\Lambda p\times q$ be denoted by $p\times q$, and let $p=p\times p$. These points $p\times q$ make up the topological product $P=K\times K$. Consider the closed segment of pq from $p\times q$ to $q\times p$, it being understood that this segment is the point p when $p=q$. All such segments form the "neighborhood" N_S. Clearly N_S can be homotopically deformed in N_S along the segments pq upon S with S remaining pointwise invariant. Finally consider the closed segment of pq from p to $p\times q$, it being understood that this segment is the point p when $p=q$. All such segments form the "neighborhood" N_K. Clearly N_K can be homotopically deformed in N_K along the segments pq upon K with K remaining pointwise invariant.

Received by the editors July 23, 1943.

[1] Numbers in brackets refer to the References at the end of the paper.

There is a natural cell division of P whose oriented cells are $\sigma \times \tau$ for all oriented simplexes σ, τ of K.[2] Let $|\sigma \times \tau|$ denote the carrier of $\sigma \times \tau$, that is, the point set of J associated with $\sigma \times \tau$. The rays pq determine a mapping Λ of P on S. The cells $\Lambda|\sigma \times \tau| = \Lambda|\tau \times \sigma|$ form a natural cell division of S.[3] With this division Λ is a cell mapping. Furthermore $\Lambda\sigma \times \tau$ is an orientation of $\Lambda|\sigma \times \tau|$ such that

$$\Lambda\sigma^s \times \tau^t = (-1)^{st}\Lambda\tau^t \times \sigma^s, \tag{1}$$

the superscripts denoting dimension.[3]

THEOREM 1. *There is a simplicial division of J such that K, S, P, N_S, and N_K carry subcomplexes of the division, the first of which is a subdivision of the given polyhedron K, and the second and third of which are subdivisions of the above natural cell divisions of S and P.*

PROOF. Let $[\sigma, \tau]$ be the point set made up of the points of all segments pq with $p \in \sigma$ and $q \in \tau$. If σ and τ have no common vertex, $[\sigma, \tau] = (\sigma, \tau)$, the join[4] of σ and τ. In this case S and P separate $[\sigma, \tau]$ into four $(s+t+1)$-cells, where s and t are the dimensions of σ and τ, respectively.

Consider $[\sigma, \sigma]$. As an auxiliary set we construct the join (σ, σ') with σ' homeomorphic to σ. Each pair of points, one in σ and one in σ', determines their join in (σ, σ'). This join of the two points is called a ray of (σ, σ'). The mid-points of the rays form a set homeomorphic to $\sigma \times \sigma'$ or $\sigma \times \sigma$. This product separates (σ, σ') into two cells. We discard the one of these cells that contains σ' and retain C, the closure of the other cell which contains σ, $\sigma \times \sigma$, and rays connecting these two sets. We subdivide $\sigma \times \sigma$ into a simplicial complex in such a way that the set of points $p \times p$, all p in σ, carries a subcomplex (that such a division exists is proved in [5]). Each simplex of this division is extended to a cell of C by adding to the simplex the points of all rays of C with one end point in the simplex (the set so obtained is a cell because it is obtained from a prism whose bases are simplexes by simple identifications in one base of the prism). These cells form a division of C into a cell complex. The rays of C that join p and $p \times p$, all p in σ, carry a subcomplex of this cell division. Let each such ray be reduced to a point by identification of all points of the ray. It is seen that the transform of C under this identification is a cell complex which is a division of $[\sigma, \sigma] \cap N_K$.

[2] The properties of P which we use can be found in [3].

[3] See [5].

[4] A discussion of the join of two complexes is found in [3]. Also the properties of the join which we assume are presented in [1].

To obtain a division of $[\sigma, \sigma]\cap N_S$ we consider the prism D defined as the product of $\sigma\times\sigma$ and a 1-simplex. We use the same simplicial division of $\sigma\times\sigma$ used in the preceding paragraph. This division is extended to a cell division of D by adding to each simplex the product of the simplex and the 1-simplex used to define D. One base of the prism D is reduced to the symmetric product of σ by the identification of $p\times q$ and $q\times p$ for all p, q in σ. If our original simplicial division of $\sigma\times\sigma$ is properly determined (as in [5]), these identifications transform D into a cell complex E.[8] Next the rays of E which are images of the rays of D with end points $p\times p$, all p in σ, are reduced each to a point by identification. It is seen that the resulting space is a cell complex which is a division of $[\sigma, \sigma]\cap N_S$.

In the same way we subdivide $[\sigma, \tau]$ where σ and τ have some but not all of their vertices in common. It is seen that Theorem 1 is true.

3. **Homologies in P and S.** Let z_i, $i=1, 2, \cdots$, be the cycles of a canonical basis[5] for the chains of all dimensions of K, and let f_i be the non-cyclic chains of this canonical basis. The range of the subscript of f is a subset of the range of the subscript of z. For each f there is a boundary relation $Ff_i^{r+1}=e_iz_i^r$, $e_i\geqq 1$. With each z_i that is free there is associated an e_i equal to zero.

A homology base for P is given by the cycles $z_i\times z_j$ and $(1/(e_i, e_j))F(f_i\times f_j)$, the denominator of the coefficient denoting the greatest common divisor of e_i and e_j. A complete set of homology relations for the cycles of this base is given by the two sets

$$(e_i, e_j)z_i\times z_j\sim 0, \qquad F(f_i\times f_j)\sim 0, \tag{2}$$

with the understanding that $(a, 0)=a$.

We next obtain a homology base for S. We consider

$$\Delta z_i^r\times z_j^s=(-1)^r\Delta z_j^s\times z_i^r, \qquad i\neq j, \tag{3}$$

the equality holding because of (1). The dimension superscripts in (3) could be r_i and r_j rather than r and s, but we use the less clear notation to simplify the printing. We also consider

$$\Delta z_i^r\times z_i^r, \qquad r \text{ even}, \tag{4}$$

$$\Delta\frac{1}{(e_i, e_j)}F(f_i^{r+1}\times f_j^{s+1})=(-1)^{(r+1)(s+1)}\Delta\frac{1}{(e_i, e_j)}F(f_j^{s+1}\times f_i^{r+1}), \qquad i\neq j, \tag{5}$$

[8] See [3, p. 104].

and

$$\frac{1}{2}\,\Lambda\,\frac{1}{e_i}F(f_i^{r+1}\times f_i^{r+1}), \qquad r \text{ odd}. \tag{6}$$

Using [6, p. 22, line 15] and the method of [2, §4] we infer that the cycles (6) are integral, the orders of (3), (4), and (5) are (e_i, e_j) while (6) have the orders $2e_i$, the cycles (3), (4), (5), and (6) are a homology base for S, and the orders of these cycles are a complete set of homology relations in S.

4. **The sum cycles and their group** S. We shall apply the theory of the sum of two complexes to $J=N_S+N_K$. To prepare for this we define a sum cycle to be a cycle that is expressible as the sum of two cycles, one in N_S and the other in N_K. The group of the sum cycles modulo those that bound in J is called S. This group S is known to be a subgroup of the Betti group of J. Here, as throughout the paper unless otherwise stated, we do not define a group for each dimension but combine all elements of all dimensions into a single group.

THEOREM 2. *The group S is generated by a free 0-dimensional element and a set of elements of order 2, each containing one and only one of the cycles* (6).

PROOF. Any cycle of K can be deformed in J onto a point p of K along the segments pq. Since N_K is a deformation retract of K, the group of the cycles of N_K modulo those that bound in J has a single nonzero 0-dimensional element as generator.

A cycle of S which bounds in J is homologous in N_S to a cycle of P. Indeed if $z=FC$, z in S, C in J, and C simplicial, then C can be expressed as the sum of two chains, one in N_S and the other in N_K. The boundary of the first of these two chains is z minus a cycle of P.

Any cycle z of P is homologous in N_S to Λz. Hence using (3), (4), (5), and (6) we obtain Theorem 2.

5. **The seams and their group** Γ. The seams are defined to be the cycles of P that bound both in N_S and N_K. We note that if z is a cycle of P and $z\sim 0$ in N_S, then since z and Λz are homotopic, $\Lambda z\sim 0$ in N_S. But N_S was shown in §1 to be a neighborhood retract of S. Hence $\Lambda z\sim 0$ in S. From this observation, from (1), and from §3 we deduce that the cycles of P that bound in N_S are the cycles homologous in P to all linear combinations of

$$(z_i^r\times z_j^s)-(-1)^{rs}(z_j^s\times z_i^r), \qquad i\neq j, \tag{7}$$

$$z_i^r \times z_i^r, \qquad r \text{ odd}, \tag{8}$$

$$\frac{1}{(e_i, e_j)}\left[F(f_i^{r+1} \times f_j^{s+1}) - (-1)^{(r+1)(s+1)} F(f_j^{s+1} \times f_i^{r+1})\right], \qquad i \neq j, \tag{9}$$

and

$$\frac{1}{e_i} F(f_i^{r+1} \times f_i^{r+1}), \qquad r \text{ even}. \tag{10}$$

To find the cycles of P that bound in N_K we consider the transformation $M(p \times q) = p$. We have $M|\sigma \times \tau| = |\sigma|$, the notation being that of §2. Also for any set $A \subset P$ the discussion of §2 implies that A and $M(A)$ are homotopic in N_K. Hence $z_i^r \times z_j^s$ is homologous in N_K to $M(z_i^r \times z_j^s)$ which is carried by $|z_i^r|$. Hence if $s > 0$, $z_i^r \times z_j^s$ bounds in N_K because it can be homotopically deformed into a set of dimension less than the dimension of the cycle.

Suppose $s = 0$. We assume that z_i^0 is equal to 1 at exactly one vertex of K and is equal to 0 at all other vertices. Then $M(z_i^r \times z_j^0) = z_i^r$ because $z_i^r \times z_j^0$ is obtained by sliding z_i^r along the rays of a cone.

If in f_j^{s+1} the exponent s is greater than 0, $(1/(e_i, e_j))F(f_i^{r+1} \times f_j^{s+1}) \sim 0$ in N_K because it is homotopic to a cycle carried by $|f_i^{r+1}|$, and the dimension of the cycle is $r+s+1$ which is greater than $r+1$. If $s = 0$, the same homology holds because $e_j = 1$.

We have shown that the cycles of P that bound in N_K are the cycles homologous in P to all linear combinations of

$$z_i^r \times z_j^s, \qquad s > 0,$$

$$z_i^r \times z_1^0 - z_i^r \times z_j^0, \qquad i \neq 1,$$

and

$$\frac{1}{(e_i, e_j)} F(f_i^{r+1} \times f_j^{s+1}).$$

It is easily seen that if one cycle of a homology class of the Betti group of P is a seam, then all cycles of the class are seams. Furthermore such classes form a subgroup of the Betti group. This subgroup of the Betti group of P made up of the homology classes whose cycles are seams is called Γ. Combining the results of this section we obtain the following theorem.

THEOREM 3. *The subgroup* Γ *is generated by the homology classes that contain* (8), (9), (10),

$$(7) \quad \textit{with.} \quad r > 0 \quad \textit{and} \quad s > 0, \tag{11}$$

and

$$(z_i^r \times z_1^0) - (z_i^r \times z_j^0) - (z_1^0 \times z_i^r) + (z_j^0 \times z_i^r), \qquad i \neq 1. \tag{12}$$

6. **The Betti group $B(J)$ of J.** The theory of the sum of two complexes[6] gives the isomorphism $B(J)/S \cong \Gamma$. However we shall obtain a sharper result.

For each seam z we shall define a pair of singular chains c of N_S and d of N_K such that $Fc = Fd = z$. To define c we deform z along the segments pq from P to S sweeping out the continuous image of a prism whose bases are $|z|$ and $|\Delta z|$. This deformation gives rise to a singular chain c whose boundary[7] is $z - \Delta z$. But since z is a seam it follows from (1), (8), (9), (10), (11), and (12) that $\Delta z = 0$. In a similar way we define the chain d of N_K.

Let $H(z)$ be the homology class of $B(J)$ which contains $c-d$. We know that any element of the Betti group $B(J)$ is expressible as the sum of an element of S and an $H(z)$, z a seam.[6] Hence to characterize $B(J)$ we need only find the relations that involve the $H(z)$. We know[6] that any homology among the $H(z)$ implies the same homology among the z. On the other hand, $z \sim 0$ in P implies[6] that $H(z)$ is in S. We shall examine all relations $z \sim 0$ in P, and find the corresponding relations among the $H(z)$. The relations to examine are (2) applied to (8), (9), (10), (11), and (12).

First we consider $(e_i, e_j)(11) \sim 0$ in P. We consider

$$
\begin{aligned}
C = {} & \alpha[(f_i^{r+1} \times z_j^s) - (-1)^{s(r+1)}(z_j^s \times f_i^{r+1})] \\
& + \beta[(-1)^r(z_i^r \times f_j^{s+1}) - (-1)^{rs}(f_j^{s+1} \times z_i^r)],
\end{aligned} \tag{13}
$$

with α, β integers such that $\alpha e_i + \beta e_j = (e_i, e_j)$. Since we find in [3, p. 138, (5.5)] that

$$
F(x^p \times y) = (Fx^p \times y) + (-1)^p(x^p \times Fy), \tag{14}
$$

it follows that $FC = (e_i, e_j)(11)$. Next we deform C along the segments pq from P to S sweeping out the continuous image of a prism whose bases are C and ΔC. This deformation gives rise to a singular chain[7] whose boundary is $\Delta C - C - (e_i, e_j)c$ because, as defined above, $(e_i, e_j)c$ is swept out by $FC = (e_i, e_j)z$. Using (1) we calculate that $\Delta C = 0$. Hence $(e_i, e_j)c + C$ bounds in N_S. In the same way it is seen that $(e_i, e_j)d + C$ bounds in N_K. Hence $(e_i, e_j)(c-d) \sim 0$ in J and $(e_i, e_j)H(z) = 0$.

In the same way we show that $e_iH(12) = (e_i, e_j)H(9) = e_iH(10) = 0$. The argument is as above with (13) replaced by

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[6] See [3, p. 267, 18].

[7] See [3, p. 307, (21.2)].

$$C = (f_i^{r+1} \times z_1^0) - (f_i^{r+1} \times z_j^0) - (z_1^0 \times f_i^{r+1}) + (z_j^0 \times f_i^{r+1}),$$

$$C = (f_i^{r+1} \times f_j^{s+1}) - (-1)^{(r+1)(s+1)}(f_j^{s+1} \times f_i^{r+1}),$$

and

$$C = (f_i^{r+1} \times f_i^{r+1}),$$

respectively.

Finally we consider $e_i(z_i^r \times z_i^r) \sim 0$ in P, r odd. Take $C = (f_i^{r+1} \times z_i^r)$. Then $FC = e_i(z_i^r \times z_i^r)$. Using (14), (1), and the fact that r is odd, we compute that

$$\Delta F(f_i^{r+1} \times f_i^{r+1}) = 2\Delta(f_i^{r+1} \times z_i^r).$$

Hence

$$e_i H(z_i^r \times z_i^r) \supset (1/2)\Delta F(f_i^{r+1} \times f_i^{r+1}).$$

This implies that the elements of S with dimension greater than zero are in the subgroup of $B(J)$ generated by the $H(z)$, z a seam. Also $H(z_i^r \times z_i^r)$, r odd, is of order $2e_i$. Our discussion implies the following:

THEOREM 4. *The Betti group $B(J)$ is generated by a nonzero 0-dimensional element, $H(8)$, $H(9)$, $H(10)$, $H(11)$, and $H(12)$; furthermore a complete set of relations for these generators consists of $2e_iH(8) = (e_i, e_j)H(9) = e_iH(10) = (e_i, e_j)H(11) = e_iH(12) = 0$.*

We shall conclude with two corollaries of Theorem 4. Let K_a be K augmented as defined in [3, p. 130]. Let $B(K_a)$ be generated by U_i, $i = 1, 2, \cdots$, and u_k, $k = 1, 2, \cdots$, where the U_i are free and independent, u_k is of order e_k, and the e_k are the torsion coefficients. Let the dimensions of U_i and u_k be r_i and r_k, respectively. Let B^n denote the n-dimensional Betti group of J.

COROLLARY 1. *To each pair U_i and U_j, $i \neq j$, $r_i + r_j = n-1$, there corresponds a free element of B^n; to each U_i, $r_i = (n-1)/2$, r_i odd, there corresponds a free element of B^n; to each pair u_k and u_l, $k \neq l$, $r_k + r_l$ = either $n-1$ or $n-2$, there corresponds an element of B^n of order (e_k, e_l); to each u_k, $r_k = (n-1)/2$, r_k odd, there corresponds an element of B^n of order $2e_k$; and to each u_k, $r_k = (n-2)/2$, r_k even, there corresponds an element of B^n of order e_k; the elements mentioned generate B^n, and their orders are a complete set of relations for these generators; finally B^0 is a free cyclic group of rank 1.*

Let R^n and $R^n(J)$ denote the n-dimensional Betti numbers of K_a and J, respectively.

COROLLARY 2. *We have*

$$R^n(J) = \sum R^i R^{n-i-1} + A^n,$$

with the summation over all integers i from 0 to the greatest integer less than $(n-1)/2$ and

$$A^n = \begin{cases} R^{(n-1)/2}, & (n-1)/2 \text{ an odd integer}, \\ 0, & (n-1)/2 \text{ not an odd integer}. \end{cases}$$

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THE PROBABILITY THAT A DETERMINANT BE CONGRUENT TO a (MOD m)

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The problem treated is the evaluation of $P_n(a, m)$, the probability that a determinant of order n having integral elements be congruent to a modulo m. By "probability" is meant the fraction obtained by dividing the number of favorable cases by the number of possible cases: let each element of the determinant range over the values $1, 2, \cdots, m$; among the m^{n^2} possible determinants let g be the number which are congruent to a modulo m; then $P_n(a, m) = g/m^{n^2}$.

This problem has been investigated by Jordan,[1] whose solution involves the function

$$S_n(p, \lambda) = \sum p^{-(\lambda_1+\lambda_2+\cdots+\lambda_{n-1})} \qquad (n \geqq 2), \tag{1}$$

where the sum ranges over all values satisfying the inequality

$$0 \leqq \lambda_1 \leqq \lambda_2 \leqq \cdots \leqq \lambda_{n-1} \leqq \lambda.$$

We use here a different method and obtain results more explicit than (1). Our results can be obtained from Jordan's, but it is as convenient to derive them independently.

It will be convenient to use a result stated by Hull,[2] which in our notation can be written

$$P_n(a, m_1)P_n(a, m_2) = P_n(a, m_1m_2) \quad \text{if} \quad (m_1, m_2) = 1. \tag{2}$$

We shall prove

$$P_n(a, m) = P_n(q, m) \quad \text{where} \quad q = (a, m), \tag{3}$$

so that our problem has been reduced to the determination of $P_n(p^\alpha, p^k)$, p being a prime. This can be evaluated by means of

$$P_n(p^\alpha, p^k) = \{\phi(p^{k-\alpha})\}^{-1}\{P_n(0, p^\alpha) - P_n(0, p^{\alpha+1})\} \qquad (0 \leqq \alpha < k), \tag{4}$$

and

$$P_n(0, p^k) = 1 - \prod_{r=k}^{k+n-1} (1 - p^{-r}) \qquad (k \geqq 1), \tag{5}$$

Presented to the Society, November 27, 1943; received by the editors July 30, 1943.

[1] C. Jordan, *Sur le nombre des solutions de la congruence* $|a_{ik}| \equiv A \pmod{M}$, J. Math. Pures Appl. (6) vol. 7 (1911) pp. 409–416.

[2] Ralph Hull, *Congruences involving kth powers*, Trans. Amer. Math. Soc. vol. 34 (1932) p. 910.

the Euler ϕ-function being involved in (4). We now prove these results.

1. **Two lemmas.** The first is due to C. Jordan.[3] Let $N_n(d, m)$ represent the number of different sets $a_1, a_2, \cdots, a_n$ of n positive integers not greater than m such that $(a_1, a_2, \cdots, a_n, m) = d$, d being any divisor of m. This, with $d = 1$, is Jordan's generalization of the Euler ϕ-function.

LEMMA 1.

$$N_n(d, m) = N_n(1, m/d);$$

$$N_n(1, ab) = N_n(1, a)N_n(1, b) \quad \textit{if} \quad (a, b) = 1;$$

$$N_n(1, p^k) = (p^k)^n - (p^{k-1})^n \quad \textit{for} \quad k \geqq 1, \quad p \textit{ any prime.}$$

The first equation reduces our function to that of Jordan, and the other equations are his.

LEMMA 2. *Let $a_1, a_2, \cdots, a_n$ be any integers such that $(a_1, a_2, \cdots, a_n, m) = 1$. Then integral solutions λ_i can be found for the congruence*

$$\sum_{i=1}^{n} a_i\lambda_i \equiv 1 \pmod{m}, \tag{6}$$

such that λ_1 is prime to m.

Let $x_1, x_2, \cdots, x_n$ be any solution of the congruence (6). Let c be defined by $c = (a_2, a_3, \cdots, a_n, m)$, so that we have $(x_1, c, m) = 1$. Also there exist integers $k_2, k_3, \cdots, k_n$ such that

$$\sum_{i=2}^{n} k_i a_i \equiv c \pmod{m}.$$

Let b be the product of those primes which divide m but not x_1. Then we set

$$\lambda_1 = x_1 + \sum_{i=2}^{n} bk_i a_i, \qquad \lambda_j = x_j - bk_j a_1 \qquad (j = 2, 3, \cdots, n),$$

noting that these give a solution of (6). We have $\lambda_1 \equiv x_1 + bc \pmod{m}$. Let p be any prime factor of m. If $p \mid x_1$, then $p \nmid b$ and $p \nmid c$. If $p \nmid x_1$, then $p \mid b$. Hence λ_1 is prime to m.

2. **A recursion formula.** Consider now the determinant of order n

[3] *Traité des substitutions*, Paris, 1870, pp. 95–97, or L. E. Dickson, *History of the theory of numbers*, vol. 1, p. 147.

with arbitrary integral elements modulo m. Let $a_1, a_2, \cdots, a_n$ and $b_1, b_2, \cdots, b_n$ designate the elements of the first two rows, respectively. We shall assume that the g.c.d. of the elements of the first row and m has been factored and placed in front of the determinant, so that $(a_1, a_2, \cdots, a_n, m) = 1$. Using Lemma 2 we choose $\lambda_1, \lambda_2, \cdots, \lambda_n$ so that (6) holds with $(\lambda_1, m) = 1$. We now apply the following transformation to the determinant. Multiply the first column by λ_1. This of course multiplies the value of the determinant by λ_1, which is immaterial since in this section we are concerned with divisibility by m, that is, with $P_n(0, m)$. Next we add to the elements of the first column the elements of the succeeding columns multiplied by $\lambda_2, \lambda_3, \cdots, \lambda_n$, respectively. Having obtained 1 in the first place, we use it to eliminate all succeeding elements of the first row (by subtracting from the ith column the elements of the first column multiplied by a_i, for $i = 2, 3, \cdots, n$), so that the first row is now $1, 0, 0, \cdots, 0$. Suppose the second row has become $c_1, c_2, \cdots, c_n$.

We propose to show that the transformation thus effected on the second row (and hence on all succeeding rows) is unique in the following sense: for any fixed set of a's in the first row and any set of c's, there is a unique set of b's which is transformed into the set of c's. The b's and c's are connected by the equations

$$c_1 = \sum_{j=1}^{n} \lambda_j b_j, \qquad c_i = b_i - a_i c_1 \qquad (i = 2, 3, \cdots, n).$$

The determinant of the transformation has the value λ_1 (prime to m), and the result follows.

Now the value modulo m of the transformed determinant is deduced from the minor of the leading element, and since some divisor (say d) of m was removed from the elements of the first row, we are interested in the probability $P_{n-1}(0, m/d)$ that m/d divide this principal minor. There are $N_n(d, m)$ arrangements of the first row having d for the g.c.d. of the elements and m, and for each of these arrangements there are m^{n^2-n} possible arrangements for all the other rows (m possibilities for each element). Among these the number of favorable cases, that is, determinants divisible by m, is

$$m^{n^2-n} N_n(d, m) P_{n-1}(0, m/d).$$

By considering all possible divisors d of m we get the total number of favorable cases. But this total number can also be obtained by multiplying the number m^{n^2} of possible determinants by $P_n(0, m)$. Hence we have

$$m^{n^2}P_n(0, m) = \sum_{d|m} m^{n^2-n}N_n(d, m)P_{n-1}(0, m/d).$$

Using the first equation in Lemma 1 and the fact that m/d ranges over the divisors of m as d does, and dividing by m^{n^2-n}, we obtain

$$m^nP_n(0, m) = \sum_{d|m} N_n(1, d)P_{n-1}(0, d) \qquad (n > 1, m \geqq 1). \tag{7}$$

If m is a power of a prime, say p^k, we have

$$p^{kn}P_n(0, p^k) = \sum_{\alpha=0}^{k} N_n(1, p^\alpha)P_{n-1}(0, p^\alpha). \tag{8}$$

3. **The formula for $P_n(0, p^k)$.** Subtracting equation (8) from the corresponding equation with k replaced by $k+1$, and using the last formula in Lemma 1, we obtain the recursion formula

$$p^{(k+1)n}P_n(0, p^{k+1}) = p^{kn}P_n(0, p^k) + (p^{(k+1)n} - p^{kn})P_{n-1}(0, p^{k+1}).$$

It is convenient here to set $Q_n^k = 1 - P_n(0, p^k)$ and $q = p^{-1}$. Making these substitutions we have

$$Q_n^{k+1} = q^nQ_n^k + (1 - q^n)Q_{n-1}^{k+1}. \tag{9}$$

We shall now prove

$$Q_n^k = \prod_{r=k}^{k+n-1} (1 - q^r) \qquad (k = 0, 1, 2, \cdots; n = 1, 2, 3, \cdots). \tag{10}$$

This result is trivial for $k = 0$ and all n, and for $n = 1$ and all k. Assume it true for $k \leqq K$ and all n, and for $k = K+1$ and $n < N$. To complete the induction we must prove it for $k = K+1$ and $n = N$. By (9) and the induction hypotheses we have

$$\begin{aligned} Q_N^{K+1} &= q^N \prod_{r=K}^{K+N-1} (1 - q^r) + (1 - q^N) \prod_{r=K+1}^{K+N-1} (1 - q^r) \\ &= \prod_{r=K+1}^{K+N} (1 - q^r). \end{aligned}$$

This proves (10) and equation (5) follows.

4. **Proofs of (3) and (4).** In the light of (2), we need demonstrate (3) for $m = p^k$ only. We wish to prove, then, that the number of determinants congruent to p^α (mod p^k) equals the number congruent to cp^α, where $(c, p) = 1$ and $\alpha < k$. Now any one of the former type gives one of the latter if the first row (say) is multiplied by c, and con-

versely if the first row is multiplied by the inverse of c (mod p^k). This inverse exists, and the correspondence is one-to-one, because c is prime to p. This proves (3).

The sum of the probabilities $P_n(ap^\alpha, p^k)$, where a runs through the values $1, 2, \cdots, p^{k-\alpha}$, is clearly the probability that a determinant be divisible by p^α. The terms of this sum can be simplified and collected by use of (3), and we have

$$P_n(0, p^\alpha) = \sum_{r=0}^{k-\alpha} \phi(p^{k-\alpha-r}) P_n(p^{\alpha+r}, p^k). \tag{11}$$

Replacing α by $\alpha+1$, and subtracting the resulting equation from (11), we arrive at (4).

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ON THE NOTION OF THE RING OF QUOTIENTS OF A PRIME IDEAL

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Let $\mathfrak{o}$ be a domain of integrity (that is, a ring with unit element and with no zero divisor not equal to 0), and let $\mathfrak{u}$ be a prime ideal in $\mathfrak{o}$. We can construct two auxiliary rings associated with $\mathfrak{u}$: the factor ring $\mathfrak{o}/\mathfrak{u}$, composed of the residue classes of elements of $\mathfrak{o}$ modulo $\mathfrak{u}$, and the ring of quotients $\mathfrak{o}_\mathfrak{u}$, composed of the fractions whose numerator and denominator belong to $\mathfrak{o}$, but whose denominators do not belong to $\mathfrak{u}$. These constructions are of paramount importance in algebraic geometry; if $\mathfrak{o}$ is the ring of a variety V, there corresponds to $\mathfrak{u}$ a subvariety U of V; $\mathfrak{o}/\mathfrak{u}$ is the ring of U, whereas the ring $\mathfrak{o}_\mathfrak{u}$ is the proper algebraic tool to investigate the neighborhood of U with respect to V.

Now, the local theory of algebraic varieties involves the consideration of rings which are not domains of integrity (this, because the completion of a local ring may introduce zero divisors). Let then $\mathfrak{o}$ be any commutative ring with unit element, and let again $\mathfrak{u}$ be a prime ideal in $\mathfrak{o}$. We may define the factor ring $\mathfrak{o}/\mathfrak{u}$ exactly in the same way as above, but we cannot so easily generalize the notion of the ring of quotients $\mathfrak{o}_\mathfrak{u}$. If there exist zero divisors outside $\mathfrak{u}$, these zero divisors cannot be used as denominators of fractions, which shows that the definition of $\mathfrak{o}_\mathfrak{u}$ cannot be extended verbatim. If we

Received by the editors September 4, 1943.

consider only fractions whose denominators are not zero divisors and do not belong to $\mathfrak{u}$, we obtain a ring $\mathfrak{o}'$; but $\mathfrak{o}'$ fails in general to have the essential property of a ring of quotients, namely, of being a local ring in the sense of Krull (that is, the non-units in $\mathfrak{o}'$ will not form an ideal). The object of this note is to construct a ring for which the essential properties of rings of quotients are preserved.

Throughout this paper we shall denote by $\mathfrak{o}$ a Noetherian ring (that is, a ring in which the maximal condition for ideals is satisfied) with a unit element. Generalizing the problem of defining the ring of quotients of a prime ideal, we take any multiplicatively closed[1] subset S of $\mathfrak{o}$ which does not contain 0 (a set is said to be multiplicatively closed if the product of any two elements of the set belongs to the set; if we are concerned with a prime ideal $\mathfrak{u}$ in $\mathfrak{o}$, we take S to be the complement of $\mathfrak{u}$ in $\mathfrak{o}$). There exists at least one primary ideal which does not meet S (otherwise, 0 would belong to S as we see at once by representing the zero ideal as an intersection of primary ideals). We shall denote by $\mathfrak{s}$ the intersection of all primary ideals in $\mathfrak{o}$ which do not meet S.

PROPOSITION 1. *Let* $\{0\}=\mathfrak{q}_1\cap\cdots\cap\mathfrak{q}_h$ *be an irredundant representation of the zero ideal in* $\mathfrak{o}$ *as an intersection of primary ideals, and let* $\mathfrak{p}_i (1\leqq i\leqq h)$ *be the associated prime ideal of* $\mathfrak{q}_i$. *Assume that* $\mathfrak{p}_i\cap S=\varnothing$ *for* $i\leqq g$, *but not for* $i>g$. *Then* $\mathfrak{s}=\mathfrak{q}_1\cap\cdots\cap\mathfrak{q}_g$.

It is clear that $\mathfrak{s}\subset\mathfrak{q}'=\mathfrak{q}_1\cap\cdots\cap\mathfrak{q}_g$. Let $\mathfrak{v}$ be any primary ideal which does not meet S; we shall prove that $\mathfrak{q}'\subset\mathfrak{v}$. Let $\mathfrak{q}''$ be the ideal $\mathfrak{q}_{g+1}\cap\cdots\cap\mathfrak{q}_h$. We have $\{0\}=\mathfrak{q}'\cap\mathfrak{q}''=\mathfrak{q}'\mathfrak{q}''\subset\mathfrak{v}$, whence $\mathfrak{q}'\subset\mathfrak{v}:\mathfrak{q}''$. Let $\mathfrak{u}$ be the associated prime ideal of $\mathfrak{v}$; since $\mathfrak{v}$ contains some power of $\mathfrak{u}$, it follows from the multiplicatively closed character of S that $\mathfrak{u}$ does not meet S. If $i>g$, the ideal $\mathfrak{p}_i$ meets S and is therefore not contained in $\mathfrak{u}$. It follows[2] that $\mathfrak{v}:\mathfrak{q}''=\mathfrak{v}$, whence $\mathfrak{q}'\subset\mathfrak{v}$. Proposition 1 is thereby proved.

LEMMA 1. *Let* $\mathfrak{p}$ *be a prime ideal in* $\mathfrak{o}$, *and let* $\mathfrak{a}$ *be an ideal contained in* $\mathfrak{p}$. *If* $\mathfrak{q}$ *is an ideal containing* $\mathfrak{a}$, *the statements "*$\mathfrak{q}$ *is primary for* $\mathfrak{p}$*" and "*$\mathfrak{q}/\mathfrak{a}$ *is primary for* $\mathfrak{p}/\mathfrak{a}$*" are equivalent.*

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[1] It was H. Grell who observed for the first time that, S being any multiplicatively closed set of nonzero divisors in a ring, it is possible to associate with S a ring of quotients, whose elements are the fractions whose denominators belong to S (Cf. H. Grell, *Beziehungen zwischen Ideale verschiedener Ringe*, Math. Ann. vol. 97 (1926) p. 510). For the properties of these rings of quotients, cf. Krull, *Idealtheorie* (Ergebnisse der Mathematik) or my paper *On the theory of local rings*, Ann. of Math. vol. 44 (1943) p. 690.

[2] Cf. van der Waerden, *Moderne Algebra*, vol. 2, chap. 12, p. 36.

Lemma 1 follows trivially from the definitions.

The zero ideal in $\mathfrak{o}/\mathfrak{s}$ is the intersection of the primary ideals $\mathfrak{q}_1/\mathfrak{s}, \cdots, \mathfrak{q}_g/\mathfrak{s}$ whose associated prime ideals are $\mathfrak{p}_1/\mathfrak{s}, \cdots, \mathfrak{p}_g/\mathfrak{s}$. Let S^* be the set of the residue classes modulo $\mathfrak{s}$ of the elements of S; then $S^* \cap (\mathfrak{p}_i/\mathfrak{s}) = \emptyset$ $(1 \leqq i \leqq g)$, which means that no element of S^* is a zero divisor in $\mathfrak{o}/\mathfrak{s}$.[3] We may therefore construct the ring of quotients $(\mathfrak{o}/\mathfrak{s})_{S^*}$ of S^* with respect to the ring $\mathfrak{o}/\mathfrak{s}$.

DEFINITION 1. *The ring $(\mathfrak{o}/\mathfrak{s})_{S^*}$ will be called the ring of quotients of S with respect to $\mathfrak{o}$. This ring will be denoted by $\mathfrak{o}_S$.*

This definition coincides with the usual one in the case where S does not contain any zero divisor. We shall now prove that the essential properties of rings of quotients in the usual sense still hold in our case.

If $\mathfrak{a}$ is an ideal in $\mathfrak{o}$, $(\mathfrak{a}+\mathfrak{s}/\mathfrak{s})\mathfrak{o}_S$ is an ideal in $\mathfrak{o}_S$ which we shall denote symbolically by $\mathfrak{a}\mathfrak{o}_S$ (in spite of the fact that $\mathfrak{o}$ is not in general a subring of $\mathfrak{o}_S$, so that we cannot multiply elements of $\mathfrak{o}$ by elements of $\mathfrak{o}_S$). If $\mathfrak{b}$ is any ideal in $\mathfrak{o}_S$, the set $\mathfrak{b} \cap (\mathfrak{o}/\mathfrak{s})$ may be written in the form $\mathfrak{a}/\mathfrak{s}$, where $\mathfrak{a}$ is an ideal in $\mathfrak{o}$ which contains $\mathfrak{s}$. We shall denote $\mathfrak{a}$ symbolically by $\mathfrak{b} \cap \mathfrak{o}$ (although $\mathfrak{b} \cap \mathfrak{o}$ is not a set theoretic intersection).

PROPOSITION 2. *If $\mathfrak{b}$ is any ideal in $\mathfrak{o}_S$, we have $\mathfrak{b} = (\mathfrak{b} \cap \mathfrak{o})\mathfrak{o}_S$.*

Since $\mathfrak{o}_S$ is a ring of quotients of $\mathfrak{o}/\mathfrak{s}$, we have $\mathfrak{b} = (\mathfrak{b} \cap (\mathfrak{o}/\mathfrak{s}))\mathfrak{o}_S$.[1] Proposition 2 follows immediately from this formula.

PROPOSITION 3. *Let $\mathfrak{p}$ be a prime ideal in $\mathfrak{o}$, and let $\mathfrak{q}$ be primary for $\mathfrak{p}$. If $\mathfrak{q}$ meets S, we have $\mathfrak{p}\mathfrak{o}_S = \mathfrak{q}\mathfrak{o}_S = \mathfrak{o}_S$. If $\mathfrak{p}$ does not meet S, $\mathfrak{p}\mathfrak{o}_S$ is prime, $\mathfrak{q}\mathfrak{o}_S$ is primary for $\mathfrak{p}\mathfrak{o}_S$ and $\mathfrak{p}\mathfrak{o}_S \cap \mathfrak{o} = \mathfrak{p}$, $\mathfrak{q}\mathfrak{o}_S \cap \mathfrak{o} = \mathfrak{q}$.*

If $\mathfrak{q}$ meets S, $\mathfrak{q}+\mathfrak{s}/\mathfrak{s}$ meets S^*, whence $\mathfrak{p}\mathfrak{o}_S = \mathfrak{q}\mathfrak{o}_S = \mathfrak{o}_S$. If $\mathfrak{p}$ does not meet S, the same holds for $\mathfrak{q}$, whence $\mathfrak{s} \subset \mathfrak{q} \subset \mathfrak{p}$. By Lemma 1, $\mathfrak{p}/\mathfrak{s}$ is prime and $\mathfrak{q}/\mathfrak{s}$ is primary for $\mathfrak{p}/\mathfrak{s}$. Furthermore, $\mathfrak{p}/\mathfrak{s}$ does not meet S^*. Proposition 3 follows therefore from the corresponding proposition which is known to hold for ordinary rings of quotients.[1] We see also that, if $\mathfrak{p}$ does not meet $\mathfrak{s}$, the formula $\mathfrak{q} \leftrightarrow \mathfrak{q}\mathfrak{o}_S$ establishes a one-to-one inclusion preserving correspondence between the primary ideals for $\mathfrak{p}$ in $\mathfrak{o}$ and the primary ideals for $\mathfrak{p}\mathfrak{o}_S$ in $\mathfrak{o}_S$.

PROPOSITION 4. *Let $\mathfrak{a} = \mathfrak{b}_1 \cap \cdots \cap \mathfrak{b}_r$ be an irredundant representation of an ideal $\mathfrak{a}$ in $\mathfrak{o}$ as an intersection of primary ideals. Let $\mathfrak{u}_i$ be the associated prime ideal of $\mathfrak{b}_i$, and assume that $\mathfrak{u}_i$ meets S for $i > s$ but not for $i \leqq s$. Then we have $\mathfrak{a}\mathfrak{o}_S = \mathfrak{b}_1\mathfrak{o}_S \cap \cdots \cap \mathfrak{b}_s\mathfrak{o}_S$, and this is an irredundant representation of $\mathfrak{a}\mathfrak{o}_S$ as intersection of primary ideals in $\mathfrak{o}_S$.*

[3] If $s^* \in S^*$, we have $\{0\} : s^*(\mathfrak{o}/\mathfrak{s}) = \{0\}$ by the result quoted in footnote 2.

It is obvious that $\mathfrak{a}\mathfrak{o}_S \subset \mathfrak{b}_1\mathfrak{o}_S \cap \cdots \cap \mathfrak{b}_s\mathfrak{o}_S$. Let conversely a be any element of $\mathfrak{b}_1\mathfrak{o}_S \cap \cdots \cap \mathfrak{b}_s\mathfrak{o}_S$. We know[3] that $\mathfrak{b}_1\mathfrak{o}_S \cap \cdots \cap \mathfrak{b}_s\mathfrak{o}_S$ is equal to $(\mathfrak{b}_1/\mathfrak{s} \cap \cdots \cap \mathfrak{b}_s/\mathfrak{s})\mathfrak{o}_S$; it follows that a may be written in the form b^*/c^*, where $b^* \in \mathfrak{b}_1/\mathfrak{s} \cap \cdots \cap \mathfrak{b}_s/\mathfrak{s}$ and $c^* \in S^*$. If $i > s$, the ideal $\mathfrak{u}_i$ has an element u_i in common with S; if m is large enough, we have $u = (\prod_{i=s+1}^{r} u_i)^m \in \mathfrak{b}_{s+1} \cap \cdots \cap \mathfrak{b}_r$. Since $u_i \in S$ $(s+1 \leq i \leq r)$, we have $u \in S$, whence $u^* \in S^*$, if u^* is the residue class of u modulo $\mathfrak{s}$. We may write $a = (u^*b^*)/(u^*c^*)$, $u^*c^* \in S^*$. Let b be any element of the residue class b^* modulo $\mathfrak{s}$; since $b^* \in \mathfrak{b}_i/\mathfrak{s}$ $(1 \leq i \leq s)$, we have $b \in \mathfrak{b}_1 \cap \cdots \cap \mathfrak{b}_s$, whence $ub \in \mathfrak{a}$ and $u^*b^* \in \mathfrak{a} + \mathfrak{s}/\mathfrak{s}$, $a \in \mathfrak{a}\mathfrak{o}_S$. It is clear that none of the ideals $\mathfrak{b}_i/\mathfrak{s}$ contains the intersection of the others; making use of a known result[4] for ordinary rings of quotients, it follows that the representation $\mathfrak{a}\mathfrak{o}_S = \mathfrak{b}_1\mathfrak{o}_S \cap \cdots \cap \mathfrak{b}_s\mathfrak{o}_S$ is irredundant.

We shall now consider more specifically the case where S is the complement of a prime ideal $\mathfrak{u}$. The ring $\mathfrak{o}_S$ will then also be denoted by $\mathfrak{o}_\mathfrak{u}$. In that case, *the ideal $\mathfrak{s}$ coincides with the intersection of all primary ideals for $\mathfrak{u}$.* In fact, the set S^* is clearly the complement of $\mathfrak{u}/\mathfrak{s}$ with respect to $\mathfrak{o}/\mathfrak{s}$; the ring $\mathfrak{o}_S$ is the ring of quotients (in the ordinary sense) of the prime ideal $\mathfrak{u}/\mathfrak{s}$ with respect to $\mathfrak{o}/\mathfrak{s}$. It follows that $\mathfrak{u}\mathfrak{o}_S$ is the ideal of non-units in $\mathfrak{o}_S$, whence $\bigcap_{n=1}^{\infty} (\mathfrak{u}/\mathfrak{s})^n \mathfrak{o}_S = \{0\}$.[4] For every n, the ideal $(\mathfrak{u}/\mathfrak{s})^n \mathfrak{o}_S \cap \mathfrak{o}/\mathfrak{s}$ is a primary ideal for $\mathfrak{u}/\mathfrak{s}$ in $\mathfrak{o}/\mathfrak{s}$; it follows that the intersection of all primary ideals for $\mathfrak{u}/\mathfrak{s}$ is the zero ideal in $\mathfrak{o}/\mathfrak{s}$. Our assertion then follows from Lemma 1.[5] At the same time, we see that $\mathfrak{o}_\mathfrak{u}$ is a local ring in the sense of Krull.

Assume now that $\mathfrak{o}$ is a semi-local ring[6] and that $\mathfrak{u}$ is one of the maximal prime ideals in $\mathfrak{o}$. Let $\bar{\mathfrak{o}}$ be the completion of $\mathfrak{o}$; there corresponds to $\mathfrak{u}$ an idempotent ϵ in $\bar{\mathfrak{o}}$. We shall prove the following results:

PROPOSITION 5. *The ring $\mathfrak{o}/\mathfrak{s}$ is isomorphic with the subring $\mathfrak{o}\epsilon$ of the ring $\bar{\mathfrak{o}}\epsilon$. This isomorphism may be extended to an isomorphism of the completion of $\mathfrak{o}_\mathfrak{u}$ with $\bar{\mathfrak{o}}\epsilon$.*

The first statement will be proved if we show that $\mathfrak{s}$ coincides with the set of elements $x \in \mathfrak{o}$ which satisfy the condition $x\epsilon = 0$. If x is any

[4] Cf. Krull, *Dimensionstheorie in Stellenringen*, J. Reine Angew. Math. vol. 179 (1938) p. 204 or my paper quoted in footnote 1.

[5] This result, together with Proposition 1 above, yields a proof of a theorem of Krull; cf. Krull, *Primidealketten in allgemeinen Ringbereiche*, Sitzungsberichte der Heidelberger Akademie, 1928, p. 7.

[6] A semi-local ring is a Noetherian ring (that is, the maximal chain condition holds in the ring) in which there exist only a finite number of maximal prime ideals. For the proofs of the results on semi-local rings which are used in this paper, cf. my paper quoted in footnote 1.

element of $\mathfrak{o}$, we may write $x=x\epsilon+x(1-\epsilon)$, and we know that $1-\epsilon\in\mathfrak{u}^n\bar{\mathfrak{o}}$ for every n. If $x\epsilon=0$, we have $x\in\mathfrak{u}^n\bar{\mathfrak{o}}\cap\mathfrak{o}=\mathfrak{u}^n$ for every n; in particular, x belongs to every primary ideal for $\mathfrak{u}$, whence $x\in\mathfrak{s}$. If $x\in\mathfrak{s}$, we have $x\in\mathfrak{u}^n$ for every n ($\mathfrak{u}^n$ is primary because $\mathfrak{u}$ is a *maximal* prime ideal), whence $x\epsilon\in\mathfrak{u}^n\bar{\mathfrak{o}}\epsilon$. Since $\mathfrak{u}\bar{\mathfrak{o}}\epsilon$ is the ideal of non-units in $\bar{\mathfrak{o}}\epsilon$, it follows that $x\epsilon=0$.

If we identify $\mathfrak{o}/\mathfrak{s}$ with $\mathfrak{o}\epsilon$, every element of S^* is a unit in $\bar{\mathfrak{o}}\epsilon$. In fact, if $y\in S$, we have $y=y\epsilon+y(1-\epsilon)$, $y(1-\epsilon)\in\mathfrak{u}\bar{\mathfrak{o}}$. If we had $y\in\mathfrak{u}\bar{\mathfrak{o}}$, we would have $y\in\mathfrak{u}\bar{\mathfrak{o}}\cap\mathfrak{o}=\mathfrak{u}$, which is not the case. It follows that $\bar{\mathfrak{o}}\epsilon$ contains the ring $\mathfrak{o}_\mathfrak{u}$. The ring $\bar{\mathfrak{o}}\epsilon$ is a complete ring with $\mathfrak{u}\bar{\mathfrak{o}}\epsilon$ as unique maximal prime ideal; it is clear that $\mathfrak{o}\epsilon$ (and, a fortiori, $\mathfrak{o}_\mathfrak{u}$) is dense in $\bar{\mathfrak{o}}\epsilon$. In order to prove that $\bar{\mathfrak{o}}\epsilon$ is the completion of $\mathfrak{o}_\mathfrak{u}$, it is sufficient to prove that $\mathfrak{o}_\mathfrak{u}$ is topologically a subspace of $\bar{\mathfrak{o}}\epsilon$. We show first that $\mathfrak{u}^n\bar{\mathfrak{o}}\epsilon\cap\mathfrak{o}\epsilon=\mathfrak{u}^n\mathfrak{o}\epsilon$ for every n. Let $x\epsilon$ ($x\in\mathfrak{o}$) be an element of $\mathfrak{u}^n\bar{\mathfrak{o}}\epsilon\cap\mathfrak{o}\epsilon$; we have $x=x\epsilon+x(1-\epsilon)\in\mathfrak{u}^n\bar{\mathfrak{o}}$, whence $x\in\mathfrak{u}^n\bar{\mathfrak{o}}\cap\mathfrak{o}=\mathfrak{u}^n$, $x\epsilon\in\mathfrak{u}^n\epsilon$. The ideal $\mathfrak{u}^n\bar{\mathfrak{o}}\cap\mathfrak{o}_\mathfrak{u}$ is equal to $((\mathfrak{u}^n\bar{\mathfrak{o}}\cap\mathfrak{o}_\mathfrak{u}\mathfrak{o})\cap\epsilon)\mathfrak{o}_\mathfrak{u}=(\mathfrak{u}^n\mathfrak{o}\epsilon)\mathfrak{o}_\mathfrak{u}=\mathfrak{u}^n\mathfrak{o}_\mathfrak{u}$; Proposition 5 is thereby completely proved.

PRINCETON UNIVERSITY

UNION-PRESERVING TRANSFORMATIONS OF SPACE

EDWARD KASNER AND JOHN DECICCO

1. **General statement.** Sophus Lie studied transformations from lineal-elements into lineal-elements, and also transformations from surface-elements into surface-elements of space. The contact group is obtained by requiring all unions to be converted into unions. Lie's fundamental theorems may be stated as follows. All the contact lineal-element transformations form the group of extended point transformations. The contact surface-element transformations which are not merely extended point transformations are defined completely by either a single directrix equation, or a pair of directrix equations. In the first case, a point corresponds to a surface; and in the second case, a point corresponds to a curve.[1]

We extend the preceding results by studying transformations in space from differential curve-elements of order n: $(x, y, z, y', z', \cdots, y^{(n)}, z^{(n)})$, where n is 2 or more, into lineal-elements (X, Y, Z, Y', Z'). An example of such a transformation arises in the problem of finding the locus of the centers of spherical curvature for an arbitrary space curve. This problem leads to a transformation from curve-elements of third order into lineal-elements.[2]

We determine the general class of union-preserving transformations by means of a directrix equation. Lie has obtained directrix equations only for contact transformations of surface-elements since there are no contact transformations of lineal-elements besides the extended point transformations. For a point-to-surface transformation, Lie's standard directrix equation is of the form $\Omega(X, Y, Z, x, y, z) = 0$. For a point-to-curve transformation, there are two standard directrix equations of the forms $\Omega_1(X, Y, Z, x, y, z) = 0$, $\Omega_2(X, Y, Z, x, y, z) = 0$. We find that any general union-preserving transformation from curve-elements of order n into lineal-elements is completely determined by our *new directrix equation*, involving derivatives, $\Omega(X, Y, Z, x, y, z, y', z', \cdots, y^{(n-2)}, z^{(n-2)}) = 0$.

In the final part of our paper, we shall prove that the only available union-preserving transformations (in the whole domain of curve-

Presented to the Society, November 27, 1943; received by the editors August 17, 1943. We have studied the two-dimensional aspects of our new theory in Proc. Nat. Acad. Sci. (1943) and Revista des Matimaticas (1943). This leads to a large extension of the Huygens theory of evolutes and involutes and Lie theory.

[1] Lie-Scheffers, *Beruhrungstransformationen.*

[2] See Bull. Amer. Math. Soc. abstract 49-9-235 by Kasner and DeCicco, *A generalized theory of contact transformations.*

elements) are firstly, the group of point transformations, and secondly, the set of union-preserving transformations from curve-elements of order n into lineal-elements, together with the extensions of these two types.

2. **The union-preserving transformations.** Any transformation from curve-elements of order n, where n is 2 or more, into lineal-elements may be defined by equations of the form

$$
\begin{aligned}
X &= X(x, y, z, p_1, q_1, \cdots, p_n, q_n),\\
Y &= Y(x, y, z, p_1, q_1, \cdots, p_n, q_n),\\
Z &= Z(x, y, z, p_1, q_1, \cdots, p_n, q_n),\\
P &= P(x, y, z, p_1, q_1, \cdots, p_n, q_n),\\
Q &= Q(x, y, z, p_1, q_1, \cdots, p_n, q_n),
\end{aligned} \tag{1}
$$

where $p_m = y^{(m)} = d^m y/dx^m$ and $q_m = z^{(m)} = d^m z/dx^m$ for $m = 1, 2, \cdots, n$.

A series of curve-elements of order n may be defined by $y = y(x)$, $z = z(x)$, $p_1 = p_1(x)$, $q_1 = q_1(x)$, $\cdots$, $p_n = p_n(x)$, $q_n = q_n(x)$. A series is a union if and only if the conditions $dy - p_1 dx = 0$, $dz - q_1 dx = 0$, $\cdots$, $dp_{n-1} - p_n dx = 0$, $dq_{n-1} - q_n dx = 0$ are satisfied.

A special type of union is the conical-union of order n. This consists of ∞^1 curve-elements of order n which have in common a fixed curve-element of order $(n-1)$. The equations of any conical-union are $x = x_0$, $y = y_0$, $z = z_0$, $p_1 = (p_1)_0$, $q_1 = (q_1)_0$, $\cdots$, $p_{n-1} = (p_{n-1})_0$, $q_{n-1} = (q_{n-1})_0$, $q_n = F(p_n)$.

Let us now consider what unions become unions under the transformation (1). These unions must satisfy the pair of Monge differential equations of order $(n+1)$

$$
\begin{aligned}
&(Y_{p_n} - PX_{p_n})p_{n+1} + (Y_{q_n} - PX_{q_n})q_{n+1}\\
&\quad + [(Y_z + p_1 Y_y + q_1 Y_z + \cdots + p_n Y_{p_{n-1}} + q_n Y_{q_{n-1}})\\
&\quad - P(X_z + p_1 X_y + q_1 X_z + \cdots + p_n X_{p_{n-1}} + q_n X_{q_{n-1}})] = 0,\\
&(Z_{p_n} - QX_{p_n})p_{n+1} + (Z_{q_n} - QX_{q_n})q_{n+1}\\
&\quad + [(Z_z + p_1 Z_y + q_1 Z_z + \cdots + p_n Z_{p_{n-1}} + q_n Z_{q_{n-1}})\\
&\quad - Q(X_z + p_1 X_y + q_1 X_z + \cdots + p_n X_{p_{n-1}} + q_n X_{q_{n-1}})] = 0.
\end{aligned} \tag{2}
$$

THEOREM 1. *All transformations from curve-elements of order n into lineal-elements may be divided into three distinct types with respect to the number of unions which are converted into unions.*[3]

[3] Kasner, *General transformation theory of differential elements*, Amer. J. Math. vol. 32 (1904) pp. 392–401.

Type 1°. *The union-preserving transformations. These carry every union into a union. The necessary and sufficient conditions for a union-preserving transformation are*

$$
\begin{aligned}
P &= \frac{Y_x + p_1 Y_y + q_1 Y_x + \cdots + p_n Y_{p_{n-1}} + q_n Y_{q_{n-1}}}{X_x + p_1 X_y + q_1 X_x + \cdots + p_n X_{p_{n-1}} + q_n X_{q_{n-1}}} \\
&= \frac{Y_{p_n}}{X_{p_n}} = \frac{Y_{q_n}}{X_{q_n}}, \\
Q &= \frac{Z_x + p_1 Z_y + q_1 Z_x + \cdots + p_n Z_{p_{n-1}} + q_n Z_{q_{n-1}}}{X_x + p_1 X_y + q_1 X_x + \cdots + p_n X_{p_{n-1}} + q_n X_{q_{n-}}} \\
&= \frac{Z_{p_n}}{X_{p_n}} = \frac{Z_{q_n}}{X_{q_n}}.
\end{aligned} \tag{3}
$$

Type 2°. *The intermediate transformations. These are not union-preserving transformations and satisfy the conditions*

$$
\begin{aligned}
&[(Y_x + p_1 Y_y + q_1 Y_x + \cdots + p_n Y_{p_{n-1}} + q_n Y_{q_{n-1}}) \\
&\qquad - P(X_x + p_1 X_y + q_1 X_x + \cdots + p_n X_{p_{n-1}} + q_n X_{q_{n-1}})] \\
&\cdot [(Z_x + p_1 Z_y + q_1 Z_x + \cdots + p_n Z_{p_{n-1}} + q_n Z_{q_{n-}}) \\
&\qquad - Q(X_x + p_1 X_y + q_1 X_x + \cdots + p_n X_{p_{n-1}} + q_n X_{q_{n-1}})]^{-1} \\
&\qquad = \frac{Y_{p_n} - PX_{p_n}}{Z_{p_n} - QX_{p_n}} = \frac{Y_{q_n} - PX_{q_n}}{Z_{q_n} - QX_{q_n}}.
\end{aligned} \tag{4}
$$

Any intermediate transformation carries the ∞^∞ *unions of a Monge differential equation of third order into unions.*

Type 3°. *The general transformations. These do not satisfy the conditions* (4). *Any such transformation converts precisely* $\infty^{2(n+1)}$ *unions into unions.*

The general Type 3° is defined by *five* arbitrary functions of $(2n+3)$ independent variables. The intermediate Type 2° is defined by *three* arbitrary functions of $(2n+3)$ independent variables. For this intermediate type, it is seen that (X, Y, Z) may be taken arbitrarily and then the functions P and Q are completely determined by the equations (4). Later we shall show that the union-preserving Type 1° is defined by *one* arbitrary function of $(2n+1)$ variables or by *three* arbitrary functions of $(2n+1)$ variables.

We prove Theorem 1. For union-preserving transformations the equations (2) must be identities in p_{n+1} and q_{n+1}. Upon setting the various coefficients equal to zero and solving for P and Q, we obtain

the conditions (3) for union-preserving transformations.

Next suppose we have the intermediate transformations so that the conditions (4) are satisfied. Then it is obvious that the equations (2) are equivalent. This shows that under any intermediate transformation, there are ∞^{∞} unions preserved. These satisfy either one of the two equivalent Monge differential equations (2).

Finally suppose the conditions (4) are not satisfied. In the first place it is seen that if the last equality of (4) is not valid, we can solve the equations (2) for p_{n+1} and q_{n+1}. Therefore there are $\infty^{2(n+1)}$ unions which become unions. Finally if the last equality of (4) is valid, then it can be shown that at each curve-element of order $(n-1)$ there are ∞^1 conical-unions which become unions. Since there are $\infty^{(2n+1)}$ curve-elements of order $(n-1)$, it follows that altogether there are $\infty^{2(n+1)}$ conical-unions which are converted into unions. Therefore we have proved that our general Type 3° carries exactly $\infty^{2(n+1)}$ unions into unions under all conditions.

3. **The degenerate union-preserving transformations.** Henceforth we shall consider only union-preserving transformations. However, there are certain degenerate union-preserving transformations which we wish to exclude from consideration. These degenerate correspondences are those where P and Q assume identically the values 0 or ∞; and also those for which there exist two functional relations between the three functions defininin (X, Y, Z).

It is found that these degenerate transformations may be divided into four cases: (a) those which convert every union into a union in a fixed plane parallel to one of the coordinate planes, (b) those which carry every union into a fixed straight line parallel to one of the coordinate axes, (c) those which carry every union into a conical-union of lineal-elements with fixed base-point, and (d) those which carry every union into a single fixed union.

In the rest of our article, whenever we speak of a union-preserving transformation we shall mean one which is not of the four preceding degenerate types.

4. **The special union-preserving transformations.** Any (nondegenerate) union-preserving transformation is said to be *special* if it carries every conical-union of order n into a conical-union of lineal-elements. Any such special transformation is given by equations of the form

$$
\begin{aligned}
X &= X(x, y, z, p_1, q_1, \cdots, p_{n-1}, q_{n-1}), \\
Y &= Y(x, y, z, p_1, q_1, \cdots, p_{n-1}, q_{n-1}), \\
Z &= Z(x, y, z, p_1, q_1, \cdots, p_{n-1}, q_{n-1}),
\end{aligned} \tag{5}
$$

$$
(5)\qquad
\begin{aligned}
P &= \frac{Y_x + p_1 Y_y + q_1 Y_z + \cdots + p_n Y_{p_{n-1}} + q_n Y_{q_{n-1}}}{X_x + p_1 X_y + q_1 X_z + \cdots + p_n X_{p_{n-1}} + q_n X_{q_{n-1}}},\\
Q &= \frac{Z_x + p_1 Z_y + q_1 Z_z + \cdots + p_n Z_{p_{n-1}} + q_n Z_{q_{n-1}}}{X_x + p_1 X_y + q_1 X_z + \cdots + p_n X_{p_{n-1}} + q_n X_{q_{n-1}}}.
\end{aligned}
$$

These special union-preserving transformations depend on three arbitrary functions of $(2n+1)$ independent variables. Thus there are $\infty^{3(2n+1)}$ special union-preserving transformations.

5. **The directrix equation of a general union-preserving transformation** T. We shall say that a union-preserving transformation T is *general* if it does not carry every conical-union of order n into a conical-union of lineal-elements. It follows that for such a general transformation T, we must have either $X_{p_n} \neq 0$, $Y_{p_n} \neq 0$, $Z_{p_n} \neq 0$, or $X_{q_n} \neq 0$, $Y_{q_n} \neq 0$, $Z_{q_n} \neq 0$, or both.

THEOREM 2. *For a general union-preserving transformation* T, *the four variables* $(p_{n-1}, q_{n-1}, p_n, q_n)$ *can be eliminated from the three functions defining* (X, Y, Z), *thus obtaining a single eliminant of the form*

$$
(6)\qquad \Omega(X, Y, Z, x, y, z, p_1, q_1, \cdots, p_{n-2}, q_{n-2}) = 0.
$$

We call this the *directrix equation* of our general union-preserving transformation T.

Let us proceed with the proof of the above result. In the first place, it is seen by (3) that p_n and q_n can be eliminated from the first and second, and also from the first and third of our equations (1) defining the general union-preserving transformation T. We therefore obtain the two relations

$$
(7)\qquad
\begin{aligned}
Y &= F(X, x, y, z, p_1, q_1, \cdots, p_{n-1}, q_{n-1}),\\
Z &= G(X, x, y, z, p_1, q_1, \cdots, p_{n-1}, q_{n-1}).
\end{aligned}
$$

At this point, we wish to introduce the following notation. Let X be any function of $(X, Y, Z, x, y, z, p_1, q_1, \cdots, p_m, q_m)$. Then the operation *prime* on X denotes the following linear differential operator

$$
(8)\qquad \mathrm{X}' = \mathrm{X}_x + p_1 \mathrm{X}_y + q_1 \mathrm{X}_z + \cdots + p_{m+1} \mathrm{X}_{p_m} + q_{m+1} \mathrm{X}_{q_m}.
$$

That is, X′ denotes the total derivative of X with respect to x while considering (X, Y, Z) as constants.

Next upon applying the conditions (3) for a union-preserving transformation, we discover that

$$
(9)\qquad P = F_X, \qquad Q = G_X, \qquad F' = 0, \qquad G' = 0.
$$

The eliminant with respect to X of the last two of the preceding equations must be identically zero. Differentiating the last two equations with respect to p_n, and also with respect to q_n, we find

$$(10)\qquad \begin{aligned} X_{p_n}F'_X + F_{p_{n-1}} &= 0, \qquad X_{q_n}F'_X + F_{q_{n-1}} = 0, \\ X_{p_n}G'_X + G_{p_{n-1}} &= 0, \qquad X_{q_n}G'_X + G_{q_{n-1}} = 0. \end{aligned}$$

From these four equations we can eliminate F_X' and G_X', thus finding the two eliminants

$$(11)\qquad X_{q_n}F_{p_{n-1}} - X_{p_n}F_{q_{n-1}} = 0, \qquad X_{q_n}G_{p_{n-1}} - X_{p_n}G_{q_{n-1}} = 0.$$

Since X_{p_n} and X_{q_n} can not both be zero, it follows from these equations that

$$(12)\qquad F_{p_{n-1}}G_{q_{n-1}} - F_{q_{n-1}}G_{p_{n-1}} = 0.$$

Observe that both F and G are not independent of p_{n-1} and q_{n-1}. For otherwise, if X is explicitly present in either F' or G', X would be independent of p_n and q_n as follows from equations (9). Hence X is not present in the last two of equations (9), and it results that both F and G are functions of X only. This is impossible since our transformation is not degenerate.

By the preceding remarks and equation (12), it follows that (7) has one and only one eliminant with respect to p_{n-1} and q_{n-1}, namely the equation (6). This completes the proof of Theorem 2.

THEOREM 3. *The directrix equation* (6) *completely determines the general union-preserving transformation* T, *and conversely.*

The general union-preserving transformations depend upon one arbitrary function of $(2n+1)$ independent variables. Thus there are $\infty^{f(2n+1)}$ general union-preserving transformations.

In the first place, it is seen that under the general union-preserving transformation T any point in the (X, Y, Z)-space corresponds to a family of ∞^∞ curves in the (x, y, z)-space, defined by a Monge equation of order $(n-2)$. These curves are defined by (6), where (X, Y, Z) are considered as constants. The derivatives of orders $(n-2)$, $(n-1)$, n of these curves are given by the equations

$$(13)\qquad \Omega = 0, \qquad \Omega' = 0, \qquad \Omega'' = 0.$$

Next any conical-union of order n in the (x, y, z)-space is converted into a union. This union is defined by the first two of the preceding equations where $(x, y, z, p_1, q_1, \cdots, p_{n-1}, q_{n-1})$ are regarded as constants. The first derivatives of this union are given by the equations

$$\Omega_X + P\Omega_Y + Q\Omega_Z = 0, \qquad \Omega'_X + P\Omega'_Y + Q\Omega'_Z = 0. \tag{14}$$

We shall prove that the functions defining (X, Y, Z, P, Q) of our general union-preserving transformation T must satisfy the equations (13) and (14). In the first place (X, Y, Z) must satisfy the directrix equation (6), since it was obtained as a result of eliminating $(p_{n-1}, q_{n-1}, p_n, q_n)$ from the first three of equations (1) defining our general union-preserving transformation T.

Applying the conditions (3) for union-preserving transformations to our directrix equation (6), we obtain the first two of equations (13) and the first of equations (14).

Finally upon applying the conditions (3) for union-preserving transformations to the second of equations (13), we obtain the last of equations (13) and also the last of equations (14).

That we can actually solve equations (13) and (14) for (X, Y, Z, P, Q) follows from the fact that the directrix equation (6) represents a three-parameter family of Monge equations of order $(n-2)$ in the (x, y, z)-space (or a Monge equation of order $(n+1)$ obtained by eliminating the parameters (X, Y, Z), which may be integrated three times).

By Theorems 2 and 3, it follows that the transformation from curve-elements of third order into lineal-elements, which is obtained from the problem of finding the locus of the center of spherical curvature to an arbitrary curve, is general union-preserving. On the other hand, the problem of finding the locus of the center of circular curvature to an arbitrary curve leads to a transformation from curve-elements of third-order into lineal-elements which is special union-preserving.

We note the analogy to Lie's theory of contact transformations of surface-elements. In his work, either a transformation is point-to-point, or else it is point-to-surface or point-to-curve, in which cases it is defined by a single directrix equation or by a pair of directrix equations. In our theory, a union-preserving transformation is special or general, in which case it is defined by a single directrix equation.

6. The union-preserving transformations of space in the domain of curve-elements. We shall now discuss the following result.[4]

[4] Kasner and DeCicco, *Transformation theory of integrable double-series of lineal-elements*, Bull. Amer. Math. Soc. vol. 46 (1940) pp. 93–100. Kasner, *Lineal element transformations of space for which normal congruences of curves are converted into normal congruences*, Duke Math. J. vol. 5 (1939) pp. 72–83. Kasner and DeCicco, *Curvature element transformations which preserve integrable fields*, Proc. Nat. Acad. Sci. U.S.A. vol. 25 (1939) pp. 104–111.

THEOREM 4. *If a union-preserving transformation T from curve-elements into lineal-elements carries any two unions which possess $n \geqq 2$ as the order of contact into two unions which have, at least, second order contact, then T must be an extended point-transformation.*

In the first place, it is obvious that our union-preserving transformation T is at most from curve-elements of order n into lineal-elements.

Let us first of all assume that T is a general union-preserving transformation. The first derivatives $P=dY/dX$ and $Q=dZ/dX$ of any transformed union under the general union-preserving transformation T defined by (13) and (14) are given by (14). We proceed to find the second derivatives $dP/dX=d^2Y/dX^2$ and $dQ/dX=d^2Z/dX^2$.

In the first place, by differentiating the last of equations (13) totally with respect to x, we find

$$(\Omega''_X + P\Omega''_Y + Q\Omega''_Z)dX/dx = -\Omega'''. \tag{15}$$

Next differentiating the equations (14) totally with respect to x and using the conditions (14) and (15), we find

$$\begin{aligned} &\Omega_Y dP/dX + \Omega_Z dQ/dX + \Omega_{XX} + P^2\Omega_{YY} + Q^2\Omega_{ZZ} \\ &\quad +2P\Omega_{XY} + 2Q\Omega_{XZ} + 2PQ\Omega_{YZ} = 0, \\ &\Omega'_Y dP/dX + \Omega'_Z dQ/dX + \Omega'_{XX} + P^2\Omega'_{YY} + Q^2\Omega'_{ZZ} \\ &\quad + 2P\Omega'_{XY} + 2Q\Omega'_{XZ} + 2PQ\Omega'_{YZ} = (1/\Omega''')(\Omega''_X + P\Omega''_Y + Q\Omega''_Z)^2. \end{aligned} \tag{16}$$

Now apply the conditions of our theorem. Then dP/dX and dQ/dX can not contain p_{n+1} and q_{n+1}. The only place where these appear are in Ω'''. Their coefficients are Ω_{p_n-2} and Ω_{q_n-2} which can not be zero. Therefore our transformation T is not a general union-preserving transformation from curve-elements of order n into lineal-elements.

Finally let us suppose that our transformation T is special union-preserving from curve-elements of order n into lineal-elements. By (5), it is found upon extending our transformation T once and imposing the condition that dP/dX and dQ/dX be independent of p_{n+1} and q_{n+1} that our special union-transformation T is a union-preserving transformation from curve-elements of order $(n-1)$, at most, into lineal-elements. This is impossible.

The preceding two contradictions lead us to the conclusion that there are no union-preserving transformations from curve-elements of order n into lineal-elements, such that it converts any two unions

which possess $n \geqq 2$ as the order of contact into two unions which have at least second order contact.

Therefore the only possible union-preserving transformations which satisfy the conditions of our theorem are those from lineal-elements into lineal-elements. According to Lie's theorem, it follows that the only such transformations are the extended point transformations. This completes the proof of Theorem 4.

From this theorem, it follows that the only union-preserving transformations from curve-elements of order n into curve-elements of order m where $n \geqq m > 1$ are the extended point transformations. *Therefore the only successful union-preserving transformations (in the whole domain of curve-elements) are firstly, the point transformations, and secondly, the union-preserving transformations from curve-elements of order n, where n is 2 or more, into lineal-elements, or the extensions of these two types.*

Thus if T is a union-preserving transformation from curve-elements of order n into curve-elements of order m where $n \geqq m \geqq 1$, then, if $n = m$, T is the extension of order m of a point-transformation; and if $n > m$, T is the extension of order $(m-1)$ of a union-preserving transformation from curve-elements of order $(n-m+1)$ into lineal-elements.

7. Alternate characterization of union-preserving transformations. Under a union-preserving transformation T, a single union in the (x, y, z)-space corresponds to a single union in the (X, Y, Z)-space, but to a single union in the (X, Y, Z)-space there correspond $\infty^{2(n-1)}$ unions in the (x, y, z)-space.

(It is noted that a point (X_0, Y_0, Z_0) is not considered to be a union of lineal-elements in the (X, Y, Z)-space, but consists of $\infty^{f(1)}$ conical-unions of lineal-elements, each such conical-union corresponding to $\infty^{2(n-1)}$ unions of the Monge equation of order $(n-2)$, defined by our directrix equation (6)).

THEOREM 5. *Any transformation from curve-elements of order n of the (x, y, z)-space into lineal-elements of the (X, Y, Z)-space, by which there correspond exactly $\infty^{2(n-1)}$ unions in the (x, y, z)-space to any single union of the (X, Y, Z)-space, is a union-preserving transformation.*

Let $Y = F(x)$, $Z = G(x)$, $P = F_X$, $Q = G_X$ be any union in the (X, Y, Z)-space. Upon applying our transformation (1) to this union, we find that the corresponding unions in the (x, y, z)-space must satisfy the differential equations

$$Y(x, y, z, p_1, q_1, \cdots, p_n, q_n) = F[X(x, y, z, p_1, q_1, \cdots, p_n, q_n)],$$

(17) $$Z(x, y, z, p_1, q_1, \cdots, p_n, q_n) = G[X(x, y, z, p_1, q_1, \cdots, p_n, q_n)],$$

$$P(x, y, z, p_1, q_1, \cdots, p_n, q_n) = F_X, Q(x, y, z, p_1, q_1, \cdots, p_n, q_n) = G_X.$$

In general, we can solve these equations for $(p_{n-1}, q_{n-1}, p_n, q_n)$ in terms of $(x, y, z, p_1, q_1, \cdots, p_{n-2}, q_{n-2})$. Thus we find that there are at most $\infty^{2(n-1)}$ unions in the (x, y, z)-space which correspond to a given union in the (X, Y, Z)-space.

We now impose the conditions that these represent exactly $\infty^{2(n-1)}$ unions. In the preceding equations, we can think of $(p_{n-1}, q_{n-1}, p_n, q_n)$ as being functions of $(x, y, z, p_1, q_1, \cdots, p_{n-2}, q_{n-2})$. We can differentiate the first equation partially with respect to $(x, y, z, p_1, q_1, \cdots, p_{n-2}, q_{n-2})$, obtaining a set of $(2n-1)$ partial equations. Multiplying these equations by $(1, p_1, q_1, \cdots, p_{n-1}, q_{n-1})$ respectively, adding the results, and using the conditions that p_n is the derivative of p_{n-1} and q_n is the derivative of q_{n-1}, we obtain an equation of the same form as the first of equations (2). Similarly operating upon the second of the preceding equations, we are led to an equation of the same form as the second of equations (2). From these equations of the form (2), we are led to the conclusion that our transformation is union-preserving since the equations must be independent of p_{n+1} and q_{n+1}. This completes the proof of Theorem 5.

As an application, consider the osculating sphere to any arbitrary curve. To a single curve there corresponds a single spherical evolute, that is, the locus of the centers of spherical curvature. But to a given spherical evolute there correspond ∞^4 spherical involutes. Thus our work may be considered to be a generalized theory of spherical evolutes and involutes in space.

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REMARKS ON TRANSITIVITIES OF BETWEENNESS

W. R. TRANSUE

This note provides lattice theoretic interpretations of the transitivities

$$T_8.\quad abc \cdot dab \cdot xcd \cdot a \neq b \rightarrow acx,$$
$$T_9.\quad abc \cdot dab \cdot xcd \cdot a \neq b \rightarrow bcx,$$
$$T_{10}.\quad abc \cdot abd \cdot xbc \cdot a \neq b \cdot b \neq c \rightarrow xbd,$$

introduced by Pitcher and Smiley.[1] It may be recalled that in a lattice the relation abc (b is between a and c) is said to hold if and only if

$$(a \cup b) \cap (b \cup c) = b = (a \cap b) \cup (b \cap c).$$

THEOREM 1. *If L is a lattice then its betweenness relation has one of the transitivities T_8 or T_9 if and only if L is linearly ordered.*

PROOF. It is obvious that T_8 and T_9 are satisfied if L is linearly ordered. To show that T_8 implies linear order, consider two elements $a, c \in L$. Suppose that a and c are not comparable, that is, that none of the relations $a=c$, $a<c$, $a>c$ holds. Then $a \neq a \cup c$, $c \neq a \cup c$. Moreover, we have

$$a\, a \cup c\, c \cdot a \cap c\, a\, a \cup c \cdot a \cup c\, c\, a \cap c \cdot a \neq a \cup c$$

and by T_8 this implies $a\ c\ a \cup c$ which, with $a\ a \cup c\ c$, implies $c = a \cup c$, contrary to our assumption that $c \neq a \cup c$. In the same way T_9 can be shown to imply linear order.

THEOREM 2. *If L is a lattice then its betweenness relation has the transitivity T_{10} if and only if L is linearly ordered or is composed of two linearly ordered systems with a common greatest element, I, and a common least element, 0.*

PROOF. It is easy to see that lattice betweenness in such a lattice has the transitivity T_{10}. Denote the two linearly ordered systems by L_1 and L_2. Then if, in the hypotheses of T_{10}, $b \neq 0$, $b \neq I$, $b \in L_1$, all the elements a, c, d, and x must belong to L_1 and the conclusion follows from the fact that T_{10} holds for linear order. If $b=0$ or $b=I$ in the hypotheses of T_{10} and if $a \in L_1$, then we must have $c \in L_2$, $d \in L_2$, $x \in L_1$ and the conclusion again follows.

Received by the editors February 26, 1943.

[1] Everett Pitcher and M. F. Smiley, *Transitivities of betweenness*, Trans. Amer. Math. Soc. vol. 52 (1942) pp. 95–114. We shall use the notations and terminology of Pitcher and Smiley.

To show that T_{10} implies that the lattice must have one of these forms, we show first that the lattice consists of a number of chains connecting 0 and I unless it is linearly ordered and then show that the number of these chains cannot be greater than two. To prove the first of these statements let us show that if a and b are two elements which are not comparable then there is a greatest element, I, and a least element, 0, and $a \cup b = I$, $a \cap b = 0$. Suppose $a \cup b \neq I$, that is, that $a \cup b \geqq x$ for every $x \in L$ fails. Then there exists $u \in L$ such that $u > a \cup b$ and

$$a\, a \cup b\, u \cdot a\, a \cup b\, b \cdot b\, a \cup b\, u \cdot a \neq a \cup b \cdot a \cup b \neq u \rightarrow b\, a \cup b\, b$$

which implies $a \cup b = b$ and $b \geqq a$, contrary to the assumption that a and b are not comparable. This shows that $a \cup b = I$, and $a \cap b = 0$ may be shown in the same way.

To show that the number of chains is not greater than two we again suppose the contrary. Then we have three elements a, b, and c, no two of which are comparable. Then from T_{10}, since $a \cup b = a \cup c = b \cup c = I$,

$$a\, I\, b \cdot a\, I\, c \cdot c\, I\, b \cdot a \neq I \cdot I \neq b \rightarrow c\, I\, c$$

which implies $c = I$, contrary to our assumption that it was not comparable with a and b. This completes the proof of Theorem 2.

Finally we include a correction to the previously quoted paper of Pitcher and Smiley.[2] It is stated (p. 113) that "Examples of semi metric spaces are easily given in which τ_3 fails."

$$\tau_3. \quad abc \cdot adb \cdot dbc \rightarrow adc.$$

That this transitivity is, in fact, present in all semi metric spaces may be seen as follows. The hypotheses of τ_3 for betweenness in a semi metric space are

$$\delta(a, b) + \delta(b, c) = \delta(a, c), \tag{1}$$

$$\delta(a, d) + \delta(d, b) = \delta(a, b), \tag{2}$$

$$\delta(d, b) + \delta(b, c) = \delta(d, c). \tag{3}$$

Substituting from (2) in (1) and using (3) we have $\delta(a, d) + \delta(d, c) = \delta(a, c)$. A consequence of this fact is that no non-modular lattice can be supplied with a semi metric so that metric and lattice betweenness are the same.

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[2] This error was noticed by the authors and is included here at their request.

CONTINUED FRACTIONS AND BOUNDED ANALYTIC FUNCTIONS

H. S. WALL

1. **Introduction.** In this paper we use the characterization given by Schur [6][1] for analytic functions bounded in the unit circle together with the Stieltjes integral representation of F. Riesz [5] for analytic functions with positive real parts, to obtain a new proof of a theorem [8] characterizing totally monotone sequences in terms of Stieltjes continued fractions. In the first place, Schur used an algorithm which he called a "continued fraction-like" algorithm. We begin by constructing from this an *actual* continued fraction algorithm, and we then characterize the class of analytic functions bounded in the unit circle in terms of this continued fraction. Next, we obtain by a simple transformation a continued fraction for functions with positive real parts.[2] This along with the above mentioned-theorem of F. Riesz leads to the theorem [8, pp. 165–166] that the sequence $\{c_p\}$ is totally monotone if and only if the power series $c_0-c_1z+c_2z^2-\cdots$ is the expansion of a continued fraction of the form

$$\frac{g_0}{1+}\ \frac{g_1z}{1+}\ \frac{(1-g_1)g_2z}{1+}\ \frac{(1-g_2)g_3z}{1+}\cdots,$$

where $g_0\geqq 0$, $0\leqq g_p\leqq 1$, $p=1, 2, 3, \cdots$.

2. **An actual continued fraction algorithm derived from the "continued fraction-like" algorithm of Schur.** The continued fraction which we shall consider is as follows:[3]

$$\alpha_0+\frac{(1-\alpha_0\bar{\alpha}_0)z}{\bar{\alpha}_0z}\ \frac{1}{-\alpha_1+}\ \frac{(1-\alpha_1\bar{\alpha}_1)z}{\bar{\alpha}_1z}\ \frac{1}{-\alpha_2+}\cdots, \tag{2.1}$$

in which the α_p are complex constants with moduli not exceeding unity, and z is a complex variable. It will be convenient to suppose that if for some p, $|\alpha_p|=1$, then the continued fraction *terminates* with the first identically vanishing partial numerator.

The pth approximant of (2.1) will be denoted by $A_p(z)/B_p(z)$, where $A_0(z)=\alpha_0$, $B_0(z)=1$, $A_1(z)=z$, $B_1(z)=\bar{\alpha}_0z$, and the other nu-

Received by the editors May 26, 1943.

[1] Numbers in brackets refer to the bibliography at the end of the paper.

[2] A special case of this was given in [9, p. 415].

[3] Hamel [2] used a somewhat different continued fraction for the purpose of characterizing analytic functions bounded in the unit circle. He was obliged to use an unconventional definition of convergence.

merators and denominators are to be computed by means of the recursion formulas

$$
\begin{aligned}
A_{2p}(z) &= \alpha_p A_{2p-1}(z) - A_{2p-2}(z), \\
B_{2p}(z) &= \alpha_p B_{2p-1}(z) - B_{2p-2}(z), \\
A_{2p+1}(z) &= \bar{\alpha}_p z A_{2p}(z) + (1 - \alpha_p\bar{\alpha}_p) z A_{2p-1}(z), \\
B_{2p+1}(z) &= \bar{\alpha}_p z B_{2p}(z) + (1 - \alpha_p\bar{\alpha}_p) z B_{2p-1}(z),
\end{aligned}
\qquad p = 1, 2, 3, \cdots . \tag{2.2}
$$

If we write

$$
\pi_p = \prod_{q=0}^{p} (1 - \alpha_q\bar{\alpha}_q),
$$

then we have the "determinant formulas"

$$
A_{2p+1}(z)B_{2p}(z) - A_{2p}(z)B_{2p+1}(z) = (-1)^p z^{p+1}\pi_p, \tag{2.3}
$$

$$
A_{2p+2}(z)B_{2p}(z) - A_{2p}(z)B_{2p+2}(z) = (-1)^p z^{p+1}\pi_p\alpha_{p+1}. \tag{2.4}
$$

These may be readily derived from the recursion formulas.

We shall now establish the following theorem:

THEOREM A. *A function $f(z)$ is analytic and has modulus not greater than unity for $|z| < 1$ if and only if it is equal to a terminating continued fraction of the form* (2.1), *or is the limit for $|z| < 1$ of the sequence of even approximants of a nonterminating continued fraction of the form* (2.1).

PROOF. Except in the case where $|\alpha_0| = 1$, so that the continued fraction is equal to the constant α_0, the moduli of the even approximants are all *less* than 1 for $|z| < 1$. In fact, consider the linear fractional transformation

$$
t = t(w) = \frac{\alpha - zw}{1 - \bar{\alpha}zw} = \alpha + \frac{(1 - \alpha\bar{\alpha})z}{\bar{\alpha}z - (1/w)}, \qquad |\alpha| < 1,
$$

of the w-plane into the t-plane, the transformation depending upon the parameter z. If $|z| < 1$, this transformation has the property that $|t| < 1$ for $|w| \leqq 1$. The same is true of the product of two or more such transformations, and inasmuch as $A_{2p}(z)/B_{2p}(z)$ is equal to the product of p such transformations applied to the point $w = \alpha_p$, we conclude that

$$
\left| \frac{A_{2p}(z)}{B_{2p}(z)} \right| < 1 \quad \text{for} \quad |z| < 1. \tag{2.5}
$$

In the case where (2.1) does not terminate, the sequence of even approximants converges uniformly for $|z| \leqq r$ for every positive constant r less than 1, and represents an analytic function with modulus not greater than 1 for $|z| < 1$. In fact, from the determinant formula (2.4) we have the relation

$$\frac{A_{2p+2}(z)}{B_{2p+2}(z)} - \frac{A_{2p}(z)}{B_{2p}(z)} = \frac{(-1)^p \pi_p \alpha_{p+1} z^{p+1}}{B_{2p+2}(z) B_{2p}(z)}. \tag{2.6}$$

It is easy to see from the recursion formulas and (2.3), (2.4) that the denominators $B_{2p}(z)$ are different from zero for $|z| \leqq 1$. Hence, the expansion in ascending powers of z of the right member of (2.6) begins with the $(p+1)$th power, or a higher power, of z. Consequently, there exists a power series $P(z) = c_0 - c_1 z + c_2 z^2 - \cdots$ which agrees term by term with the series for $A_{2p}(z)/B_{2p}(z)$ for more and more terms as p is increased. Inasmuch as the coefficients in the expansion of this rational function do not exceed 1 in numerical value by virtue of (2.5), we conclude that $|c_p| \leqq 1$, $p = 0, 1, 2, \cdots$, so that $P(z)$ converges for $|z| < 1$. Moreover, if we put $A_{2p}(z)/B_{2p}(z) = a_0 - a_1 z + a_2 z^2 - \cdots$, then if $|z| \leqq r < 1$,

$$\left| P(z) - \frac{A_{2p}(z)}{B_{2p}(z)} \right| = \left| \sum_{q=0}^{\infty} (-1)^q (c_q - a_q) z^q \right| \leqq \frac{2r^{p+1}}{1-r}, \tag{2.7}$$

from which we conclude that the sequence $\{A_{2p}(z)/B_{2p}(z)\}$ converges uniformly to $P(z)$ for $|z| \leqq r < 1$; and from (2.5) it follows that $|P(z)| \leqq 1$ for $|z| < 1$.

We must now show, conversely, that if $P(z) = c_0 - c_1 z + \cdots$ is any function which is analytic and has modulus not greater than unity for $|z| < 1$, then there exists a continued fraction of the form (2.1) such that (2.7) holds. To do this, we define the function $P_1(z)$ by

$$P_1(z) = \frac{1}{z} \frac{c_0 - P(z)}{1 - \bar{c}_0 P(z)} = \frac{c_1 - c_2 z + c_3 z^2 - \cdots}{1 - \bar{c}_0 (c_0 - c_1 z + c_2 z^2 - \cdots)}.$$

It is clear that $|c_0| \leqq 1$, being the value of $P(0)$, and that if $|c_0| = 1$, then $P(z) \equiv c_0$. In either case, we put $\alpha_0 = c_0$. If we suppose that $|c_0| < 1$, then, from the character of the above transformation and from Schwarz's lemma, it follows that $P_1(z)$ is analytic and has modulus not exceeding unity for $|z| < 1$. Take $\alpha_1 = P_1(0)$. If $|\alpha_1| = 1$, then $P_1(z) \equiv \alpha_1$. If, however, $|\alpha_1| < 1$, we write

$$P_2(z) = \frac{1}{z} \frac{\alpha_1 - P_1(z)}{1 - \bar{\alpha}_1 P_1(z)}, \qquad \alpha_2 = P_2(0).$$

The function $P_2(z)$ is analytic and has modulus not greater than unity for $|z|<1$. If $|\alpha_2|=1$, then $P_2(z)\equiv\alpha_2$, while if $|\alpha_2|<1$, the process may be continued. In this way we obtain a finite or infinite sequence of functions

$$P_0(z)=P(z),\ P_1(z),\ P_2(z),\ \cdots, \tag{2.8}$$

satisfying the relations

$$P_{k+1}(z)=\frac{1}{z}\,\frac{\alpha_k-P_k}{1-\bar{\alpha}_kP_k},\qquad P_k=\frac{\alpha_k-zP_{k+1}}{1-\bar{\alpha}_kzP_{k+1}},\qquad \alpha_k=P_k(0). \tag{2.9}$$

From these we derive immediately a formal terminating or nonterminating continued fraction expansion (2.1) for $P(z)$. In the terminating case, the expansion is obviously valid, being in fact an identity. In the nonterminating case we have, for arbitrary k, the identity

$$P(z)=\frac{P_k(z)A_{2k-1}(z)-A_{2k-2}(z)}{P_k(z)B_{2k-1}(z)-B_{2k-2}(z)},$$

so that, by (2.3),

$$P(z)-\frac{A_{2k-2}(z)}{B_{2k-2}(z)}=\frac{P_k(z)(-1)^{k-1}\pi_{k-1}z^k}{B_{2k}(z)[P_k(z)B_{2k-1}(z)-B_{2k-2}(z)]}.$$

From this it readily follows that the power series for $A_{2k-2}(z)/B_{2k-2}(z)$ agrees term by term with $P(z)$ for more and more terms as k is increased, and consequently (2.7) holds.

This completes the proof of Theorem A. This proof is the same as that of Schur [6], except that we have used the formulas and notation of continued fractions. It should be observed that the constants α_ν are uniquely determined by means of the given function $P(z)$, either as an infinite sequence in the one case or as a finite sequence in the other.

3. **A continued fraction expansion for functions with positive real parts.** A function $k(z)$ is analytic and has a nonnegative real part for $|z|<1$ and has the value 1 for $z=0$ if and only if there exists a function $P(z)$ which is analytic and has modulus not greater than 1 for $|z|<1$, such that

$$k(z)=\frac{1+zP(z)}{1-zP(z)}. \tag{3.1}$$

This follows from Schwarz's lemma and the relation

$$R(k(z)) = \frac{1 - |zP(z)|^2}{|1 - zP(z)|^2}.$$

We shall now obtain a continued fraction expansion for $k(z)$.

Let $\beta_0 = 1$, and determine $\beta_1, \beta_2, \cdots$ by the relations

$$\beta_{p+1} = -\frac{\bar{\alpha}_p - \beta_p}{1 - \alpha_p\beta_p}, \qquad p = 0, 1, 2, \cdots, \tag{3.2}$$

where the α_p are the numbers appearing in the continued fraction (2.1) for the function $P(z)$ determined by (3.1). The numbers β_p have moduli equal to 1, and form an infinite sequence or a finite sequence according as the sequence $\{\alpha_p\}$ is infinite or finite, respectively. Instead of the functions (2.9) we now introduce functions $h_p(z)$ by

$$h_p(z) = \frac{1 - \beta_p P_p(z)}{1 + z\beta_p P_p(z)}, \qquad p = 0, 1, 2, \cdots. \tag{3.3}$$

By means of (2.9) and (3.2) we then find that

$$h_p(z) = \frac{\beta_p - \bar{\alpha}_p}{\beta_{p+1} - \beta_p z + (\beta_{p+1} + \bar{\alpha}_p)z h_{p+1}(z)}, \qquad p = 0, 1, 2, \cdots. \tag{3.4}$$

Remembering that $\beta_0 = 1$, we therefore have the formal continued fraction expansion:

$$\frac{1 - P(z)}{1 + zP(z)} = \frac{\beta_0 - \bar{\alpha}_0}{\beta_1 - \beta_0 z} + \frac{(\beta_1 + \bar{\alpha}_0)(\beta_1 - \bar{\alpha}_1)z}{\beta_2 - \beta_1 z} + \frac{(\beta_2 + \bar{\alpha}_1)(\beta_2 - \bar{\alpha}_2)z}{\beta_3 - \beta_2 z} + \cdots. \tag{3.5}$$

On multiplying both members of (3.5) by $2z/(1-z)$, adding 1 to both members, and then taking reciprocals, we have, using (3.1),

$$k(z) = \frac{1 + z}{1 - z} + \frac{2(\beta_0 - \bar{\alpha}_0)z}{\beta_1 - \beta_0 z} + \frac{(\beta_1 + \bar{\alpha}_0)(\beta_1 - \bar{\alpha}_1)z}{\beta_2 - \beta_1 z} + \frac{(\beta_2 + \bar{\alpha}_1)(\beta_2 - \bar{\alpha}_2)z}{\beta_3 - \beta_2 z} + \cdots. \tag{3.6}$$

In case $|\alpha_p| < 1$, $p = 0, 1, 2, \cdots, n-1$, $|\alpha_n| = 1$, this continued fraction terminates, the last partial quotient being equal to

$$\frac{(\beta_n + \bar{\alpha}_{n-1})(1 - \alpha_n\beta_n)z}{1 + \alpha_n\beta_n z};$$

while if $|\alpha_p|<1$, $p=0, 1, 2, \cdots$, the continued fraction does not terminate. In the first case, $k(z)$ is of course equal to the continued fraction. In the second case, the continued fraction converges uniformly in the neighborhood of the origin by a well known theorem [4, p. 259]. An easy argument (cf. [9, pp. 415–416]) then shows that its value is $k(z)$.

If, in particular, $k(z)$ is real when z is real, then $\beta_p=1$, $p=0, 1, 2, \cdots$, the α_p are real, and the continued fraction for $k(z)$ can be thrown by means of an equivalence transformation into the form

$$k(z)=\frac{1+z}{1-z}\cdot\cfrac{1}{1+\cfrac{g_1w}{1+\cfrac{(1-g_1)g_2w}{1+\cfrac{(1-g_2)g_3w}{1+\cdots}}}} \tag{3.7}$$

where $w=4z/(1-z)^2$ and $g_p=(1-\alpha_{p-1})/2$, $p=1, 2, 3, \cdots$. Thus $0\leqq g_p\leqq 1$, the continued fraction terminating in case equality holds for some value of p.

It is readily seen that, conversely, if the α_p are given, then the function $k(z)$ given by (3.6) or (3.7) has the stated properties. We therefore have the following theorem:

THEOREM B. *A function $k(z)$ is analytic and has a nonnegative real part for $|z|<1$, and is equal to 1 for $z=0$, if and only if it has a continued fraction expansion of the form* (3.6), *where the α_p are constants with moduli not greater than* 1, *and the β_p are given in terms of the α_p by* (3.2), *β_0 being equal to* 1. *If $k(z)$ is real when z is real, the continued fraction can be thrown into the form* (3.7).

4. A characterization of totally monotone sequences in terms of continued fractions. A sequence $\{c_p\}$ of real numbers is called *totally monotone* if all the differences $\Delta^m c_n=c_n-C_{m,1}c_{n+1}+C_{m,2}c_{n+2}-\cdots+(-1)^m C_{m,m}c_{n+m}$, $m, n=0, 1, 2, \cdots$, are nonnegative. If $c_0=0$, then $c_p=0$ for $p=1, 2, 3, \cdots$. Excepting in this trivial case, we may normalize by dividing every member of the sequence by c_0, and may thus assume that $c_0=1$.

Hausdorff [3] showed that $\{c_p\}$ is totally monotone if and only if there exists a bounded nondecreasing function $\phi(u)$ such that

$$c_p=\int_0^1 u^p\,d\phi(u), \qquad p=0, 1, 2, \cdots. \tag{4.1}$$

This is equivalent to saying that

$$c_0-c_1w+c_2w^2-\cdots=\int_0^1\frac{d\phi(u)}{1+wu}\cdot \tag{4.2}$$

We now turn our attention to a theorem of F. Riesz [5] which, along with Theorem B, will furnish a characterization of totally monotone sequences in terms of continued fractions.

THEOREM C. *A function $k(z)$ is analytic and has a nonnegative real part for $|z|<1$, and is equal to 1 for $z=0$, if and only if it has an integral representation of the form*

$$k(z) = \int_0^{2\pi} \frac{e^{it}+z}{e^{it}-z}\,d\alpha(t), \tag{4.3}$$

where $\alpha(t)$ is a nondecreasing function such that $\alpha(0)=0$, $\alpha(2\pi)=1$. The function $\alpha(t)$ is determined uniquely to an additive constant at all its points of continuity by $k(z)$.

PROOF. Put

$$\begin{aligned} k(z) &= 1 + a_1 z + a_2 z^2 + \cdots \\ &= 1 + \sum_{p=1}^{\infty} (b_p + ic_p) r^p (\cos p\theta + i \sin p\theta), \end{aligned} \tag{4.4}$$

where $z=re^{i\theta}$, $0<r<1$. Then, if

$$u(r,\theta) = \Re(k(z)) = 1 + \sum_{p=1}^{\infty} r^p (b_p \cos p\theta - c_p \sin p\theta) \geqq 0,$$

we conclude that the function

$$\alpha_r(t) = \frac{1}{2\pi}\int_0^t u(r,\theta)\,d\theta, \qquad 0 \leqq t \leqq 2\pi,$$

is a nondecreasing function of t; and $\alpha_r(0)=0$, $\alpha_r(2\pi)=1$. Moreover,

$$\int_0^{2\pi} d\alpha_r(t) = 1, \qquad \int_0^{2\pi} 2\cos qt\,d\alpha_r(t) = r^q b_q,$$
$$\int_0^{2\pi} -2\sin qt\,d\alpha_r(t) = r^q c_q.$$

Using a well known theorem, we may now determine a nondecreasing function $\alpha(t)$ such that $\alpha(0)=0$, $\alpha(2\pi)=1$, and a sequence of values of r approaching 1, such that these equations go over into

$$\int_0^{2\pi} d\alpha(t) = 1, \qquad \int_0^{2\pi} 2\cos qt\,d\alpha(t) = b_q,$$
$$\int_0^{2\pi} -2\sin qt\,d\alpha(t) = c_q.$$

When these values are substituted in (4.4), that series may be written as the integral of a geometric series. On summing the latter, we obtain (4.3). The essential uniqueness of the function $\alpha(t)$ follows from the fact that an arbitrary continuous function can be approximated uniformly by a trigonometric polynomial [5, pp. 38–39].

Conversely, if $\alpha(t)$ is any bounded nondecreasing function such that $\alpha(0)=0$, $\alpha(2\pi)=1$, then one may verify at once that the function $k(z)$ defined by (4.3) has the required properties.

We now make in (4.3) the change of variable $u=\sin^2(t/2)$, and that formula becomes

$$(4.5)\quad k(z)=\frac{1+z}{1-z}\int_0^1\frac{d\phi(u)}{1+wu}-iw\int_0^1\frac{[u(1-u)]^{1/2}\cdot d\psi(u)}{1+wu},$$

where $\alpha(t)+\alpha(2\pi-t)=\psi(u)$, $1-\alpha(2\pi-t)=\phi(u)$, $w=4z/(1-z)^2$. The function $\psi(u)$ is of bounded variation, and $\phi(u)$ is bounded and nondecreasing: $\phi(0)=0$, $\phi(1)=1$. The second integral in (4.5) vanishes identically if and only if $k(z)$ is real when z is real. On the other hand, $k(z)$ is of this character if and only if it has an expansion of the form (3.7). On putting these two facts together we obtain the relation

$$(4.6)\quad \int_0^1\frac{d\phi(u)}{1+wu}=\frac{1}{1}+\frac{g_1w}{1}+\frac{(1-g_1)g_2w}{1}+\frac{(1-g_2)g_3w}{1}+\cdots$$

Recalling now the definition of a totally monotone sequence, and the statement made concerning (4.2), we have the following theorem:

THEOREM D. *The sequence $\{c_p\}$ of real numbers, of which $c_0=1$, is totally monotone if and only if the power series $c_0-c_1w+c_2w^2-\cdots$ has a continued fraction expansion of the form*

$$(4.7)\quad \frac{1}{1}+\frac{g_1w}{1}+\frac{(1-g_1)g_2w}{1}+\frac{(1-g_2)g_3w}{1}+\cdots$$

where $0\leqq g_p\leqq 1$, $p=1, 2, 3, \cdots$, it being agreed that the continued fraction shall terminate in case some partial numerator vanishes identically.

An interesting consequence of this theorem is the fact that there exists a sequence of polynomials $G_1(g_1)$, $G_2(g_1, g_2)$, $G_3(g_1, g_2, g_3)$, $\cdots$, where G_p depends upon p variables, such that every totally monotone sequence with $c_0=1$ can be represented parametrically in the form

$$c_p=G_p(g_1, g_2, \cdots, g_p), \qquad p=0, 1, 2, \cdots (G_0=1),$$

where $0\leqq g_p\leqq 1$, $p=1, 2, 3, \cdots$ Conversely, every sequence of this

form is totally monotone. To obtain the polynomials G_p it is but necessary to observe that the power series expansion of (4.7) is the series $c_0-c_1w+c_2w^2+\cdots$ where the c_p are polynomials in the g_p. These may be most conveniently calculated by means of formulas given by Stieltjes [7, pp. 419–420]. We mention only that

$$G_1(g_1) = g_1, \qquad G_2(g_1, g_2) = g_1^2(1 - g_2) + g_1g_2. \tag{4.8}$$

F. Riesz [5, p. 56] showed that a sequence $\{c_p\}$, where $c_0=1$, is totally monotone if and only if for every n the point $(c_0, c_1, \cdots, c_{n-1})$ lies in the convex extension of the continuous curve given in parametric form by $(1, u, u^2, \cdots, u^{n-1})$, $0 \leqq u \leqq 1$. The polynomials G_p therefore furnish a parametric representation for the points of this convex set. This can be verified for $n=3$ by means of (4.8).

5. **Continued fraction transformations.** If the α_p are real in (3.5), that relation may be written in the form

$$\frac{(1-z)}{2}\,\frac{1-P(z)}{1+zP(z)} = \frac{g_1}{1+}\,\frac{(1-g_1)g_2w}{1}\,+\,\frac{(1-g_2)g_3w}{1}\,+\cdots \tag{5.1}$$

where $g_p=(1-\alpha_{p-1})/2$, $p=1, 2, 3, \cdots$, and $w=4z/(1-z)^2$. To indicate the dependence of $P(z)$ upon the α_p we shall now write $P(z)=(z; \alpha_0, \alpha_1, \alpha_2, \cdots)$. The following relations may be readily verified [6]:

$$-P(z) = (z; -\alpha_0, -\alpha_1, -\alpha_2, \cdots), \tag{5.2}$$

$$P(-z) = (z; \alpha_0, -\alpha_1, \alpha_2, -\alpha_3, \cdots), \tag{5.3}$$

$$P(z^n) = (z; \alpha_0, 0, \cdots, 0, \alpha_1, 0, \cdots, 0, \alpha_2, 0, \cdots), \tag{5.4}$$

where 0 occurs $n-1$ times between α_p and α_{p+1}. We observe that the effect of replacing α_{p-1} by $-\alpha_{p-1}$ is to replace g_p by $1-g_p$; and that the effect of replacing α_{p-1} by 0 is to replace g_p by $1/2$. These facts together with the preceding relations enable us to obtain a number of transformations of the continued fraction in (5.1).

Let us denote the right member of (5.1) by $F(w)$, and write $F(w)=[w; g_1, g_2, g_3, \cdots]$. On replacing $P(z)$ by $-P(z)$ and α_{p-1} by $-\alpha_{p-1}$, that is, g_p by $1-g_p$, $p=1, 2, 3, \cdots$, we obtain at once the relation [8, p. 166; 9, p. 416]

$$\frac{1-F(w)}{1+wF(w)} = [w; 1-g_1, 1-g_2, 1-g_3, \cdots]. \tag{5.5}$$

Similarly, on replacing $P(z)$ by $P(-z)$ and α_p by $(-1)^p\alpha_p$ we get [1, p. 191]

$$1 - F(-w/(1+w)) = [w; 1 - g_1, g_2, 1 - g_3, g_4, \cdots]. \tag{5.6}$$

Using (5.4), replacing $f(z)$ by $f(z^n)$ in (5.1), we obtain a relation of the form

$$\frac{F(U_n(w))}{V_n(w) + W_n(w)F(U_n(w))} = [w; g_1, 1/2, \cdots, 1/2, g_2, 1/2, \cdots, 1/2, g_3, 1/2, \cdots], \tag{5.7}$$

where $1/2$ appears $n-1$ times in each place, and where $U_n(w)$, $V_n(w)$, $W_n(w)$ are rational functions of w given by

$$U_n(w) = \frac{4w^n}{\{((1+w)^{1/2}+1)^n - ((1+w)^{1/2}-1)^n\}^2},$$

$$V_n(w) = (1+w)^{1/2}\frac{\{((1+w)^{1/2}+1)^n - ((1+w)^{1/2}-1)^n\}^2}{((1+w)^{1/2}+1)^{2n} - ((1+w)^{1/2}-1)^{2n}},$$

$$W_n(w) = \frac{w\{((1+w)^{1/2}+1)^{n-1} - ((1+w)^{1/2}-1)^{n-1}\}^2}{2w^{n-1} - \{((1+w)^{1/2}+1)^{2n-1} - ((1+w)^{1/2}-1)^{2n-1}\}}.$$

Here, that branch of $(1+w)^{1/2}$ is to be taken which reduces to 1 for $w=0$.

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ON MINIMUM CIRCUMSCRIBED POLYGONS

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This paper contains the proofs of two theorems on the n-gon M_n of minimum area circumscribed about a convex region R in the plane. Theorem 1 shows that the area of M_n is a convex function of n and Theorem 3 shows that if R is symmetric about a point there exists an M_{2n} which is also symmetric. The corresponding theorems on inscribed polygons are also given. These theorems were conjectured by R. B. Kershner.

The symbols a and b with subscripts will be used to represent the sides of circumscribed polygons or the vertices of inscribed polygons. It will be convenient to replace the circular order of the sides (vertices) $a_0, a_1, \cdots, a_{n-1}$ of a polygon by an artificial linear ordering $a_0<a_1<\cdots<a_{n-1}<a_n$, where a_n represents the same side (vertex) as a_0. The order of the sides (vertices) of circumscribed (inscribed) polygons is established by the order of the contact points (vertices) on the boundary of the convex region.

LEMMA 1. *Let $A_n=a_0a_1\cdots a_{n-1}$ and $B_m=b_0b_1\cdots b_{m-1}$ be two polygons circumscribed about the convex region R and let either* (1) $a_0\leqq b_0<b_1<a_1\leqq a_{r-1}\leqq b_{s-1}<b_s<a_r$ *or* (2) $a_0\leqq b_0<b_1<a_1\leqq b_{s-1}\leqq a_{r-1}<a_r<b_s$. *Let $C_{n-r+s}=a_0b_1b_2\cdots b_{s-1}a_r\cdots a_{n-1}$ and $D_{m-s+r}=b_0a_1a_2\cdots a_{r-1}b_s\cdots b_{m-1}$. Then the areas satisfy the inequality $A+B\geqq C+D$ and there is equality if and only if $a_0=b_0$ and $a_{r-1}=b_{s-1}$.*

PROOF. The common part of A and B is the common part of C and D. The remaining part of $A+B$ is the remaining part of $C+D$ together with the areas of the quadrilaterals $a_0a_1b_0b_1$ and $a_{r-1}a_rb_{s-1}b_s$. Equality holds if and only if both these areas are zero, that is, if $a_0=b_0$ and $a_{r-1}=b_{s-1}$.

LEMMA 2. *If M_n is an n-gon of maximum area inscribed in the convex region R, the vertices of M_n are contact points of the sides of a circumscribed polygon.*

PROOF. If $M_n=a_0a_1\cdots a_{n-1}$, the line through a_1 parallel to the line a_0a_2 is a supporting line of R, for otherwise M_n would not have maximum area. Supporting lines determined similarly at all vertices of M_n are seen to form a circumscribed polygon.

LEMMA 3. *Let $A_n=a_0a_1\cdots a_n$ and $B_m=b_0b_1\cdots b_m$ be two polygons*

Received by the editors August 18, 1943.

inscribed in the convex region R, each having as vertices points of contact of the sides of some circumscribed polygon, and let either (1) $a_0 \leqq b_0 < b_1 < a_1 \leqq a_{r-1} \leqq b_{s-1} < b_s < a_r$ *or* (2) $a_0 \leqq b_0 < b_1 < a_1 \leqq b_{s-1} \leqq a_{r-1} < a_r < b_s$. *Let* $C_{n-r+s} = a_0 b_1 b_2 \cdots b_{s-1} a_r \cdots a_{n-1}$ *and* $D_{n-s+r} = b_0 a_1 a_2 \cdots a_{r-1} b_s \cdots b_{n-1}$. *Then the areas satisfy the inequality* $A+B \leqq C+D$ *and there is equality if and only if* $a_0 = b_0$ *and* $a_{r-1} = b_{s-1}$.

PROOF. The vertices b_0 and b_1 are in or on the triangle consisting of a_0a_1 and the adjacent sides of the circumscribed polygon corresponding to a_0 and a_1. Hence (if $a_0 \neq b_0$) a_0b_0 produced meets a_1b_1 produced. Hence $\Delta a_0b_0b_1 \leqq \Delta a_0b_0a_1$, with equality only if $a_0 = b_0$. Let a_0b_1 intersect a_1b_0 at p. Then $\Delta a_0pa_1 \geqq \Delta b_0pb_1$, with equality only if $a_0 = b_0 = p$.

Let $a_{r-1}b_s$ intersect a_rb_{s-1} at q. Then similarly, if $a_{r-1} < b_{s-1} < b_s < a_r$, $\Delta a_{r-1}qa_r > \Delta b_{s-1}qb_s$, and if $b_{s-1} < a_{r-1} < a_r < b_s$, $\Delta b_{s-1}qb_s > \Delta a_{r-1}qa_r$.

If $a_{r-1} \leqq b_{s-1} < b_s < a_r$, $C + D - A - B = \Delta a_0pa_1 - \Delta b_0pb_1 + \Delta a_{r-1}qa_r - \Delta b_{s-1}qb_s \geqq 0$ and equality holds only if $a_0 = p = b_0$ and $a_{r-1} = q = b_{s-1}$. If $b_{s-1} \leqq a_{r-1} < a_r < b_s$, $C+D-A-B = \Delta a_0pa_1 - \Delta b_0pb_1 + \Delta b_{s-1}qb_s - \Delta a_{r-1}qa_r \geqq 0$ and equality holds only if $a_0 = p = b_0$ and $a_{r-1} = q = b_{s-1}$. Thus in either case $A+B \leqq C+D$ with equality if and only if $a_0 = b_0$ and $a_{r-1} = b_{s-1}$.

THEOREM 1. *If* M_n *is an n-gon of minimum area circumscribed about the convex region R, the areas satisfy the inequality* $M_n + M_{n+2} \geqq 2M_{n+1}$.

PROOF. Let $M_n = a_0a_1 \cdots a_{n-1}$, $M_{n+2} = b_0b_1 \cdots b_{n+1}$.

Case 1. Let there be a sequence of sides $a_0 \leqq b_0 < b_1 < b_2 < a_1$. In this case, if $C_{n+1} = a_0b_1a_1 \cdots a_{n-1}$, $D_{n+1} = b_0b_2 \cdots b_{n+1}$, $M_n + M_{n+2} - C - D = a_0a_1b_0b_1 + b_0a_1b_1b_2 > 0$. Hence $M_n + M_{n+2} > C+D \geqq 2M_{n+1}$.

Case 2. Let there be no such sequence of sides. Then there must be a sequence of the form $a_0 \leqq b_0 < b_1 < a_1 \leqq a_{r-1} \leqq b_{s-1} < b_s < a_r$ since there are n intervals $[a_i, a_{i+1})$ and $n+2$ b's in these intervals. We may assume $s \leqq r+1$ since otherwise we could renumber the sides starting from a_{r-1} which we would call a_0. We may even assume that $s = r+1$, for otherwise $b_{r+1} > a_r$ and $b_{n+1} < a_n$ and hence there is a first subscript $t > r$ for which $b_{t+1} < a_t$. Then $b_t \geqq a_{t-1}$ and we may replace t by r and $t+1$ by s so that $s = r+1$. Let $C_{n+1} = a_0b_1 \cdots b_{s-1}a_r \cdots a_{n-1}$ and $D_{n+1} = b_0a_1 \cdots a_{r-1}b_s \cdots b_{n+1}$. Then by Lemma 1, $M_n + M_{n+2} \geqq C+D \geqq 2M_{n+1}$.

THEOREM 2. *If* M_n *is an n-gon of maximum area inscribed in the convex region R, the areas satisfy the inequality* $M_n + M_{n+2} \leqq 2M_{n+1}$.

PROOF. Let $M_n = a_0a_1 \cdots a_n$, and $M_{n+2} = b_0b_1 \cdots b_{n+1}$.

Case 1. Let there be a sequence of sides $a_0 \leqq b_0 < b_1 < b_2 < a_1$. In this case, if $C_{n+1} = a_0 b_1 a_1 \cdots a_{n-1}$ and $D_{n+1} = b_0 b_2 \cdots b_{n-1}$, $C+D-M_n - M_{n+2} = a_0 b_1 a_1 - b_0 b_1 b_2$ which is greater than 0 as can be shown using Lemma 2 and elementary geometry. Hence $M_n + M_{n+2} < C+D \leqq 2M_{n+1}$.

Case 2. Let there be no such sequence of sides. The proof is similar to that of Case 2 of Theorem 1 except that Lemma 3 is used instead of Lemma 1.

THEOREM 3. *If R is a convex region symmetric about a point* 0, *among the $2n$-gons of minimum area circumscribed about R there is one which is symmetric about* 0.

PROOF. Let $M_{2n} = a_0 a_1 \cdots a_{2n-1}$ be a minimum circumscribed $2n$-gon and let $M' = b_0 b_1 \cdots b_{2n-1}$ where b_i is the image in 0 of $a_{i \pm n}$. If M_{2n} is not symmetric there is a side $a_i \neq b_i$. We may assume $a_0 \neq b_0$. By reordering the sides if necessary we may assume $a_0 < b_0$ and hence $b_n < a_n$. Let t be the largest number, $0 \leqq t < n$, such that $a_t \leqq b_t$. Then $b_{t+1} < a_{t+1}$ and if we renumber starting from a_t, which we shall call a_0, we have $a_0 \leqq b_0 < b_1 < a_1$. Since a_0 and a_1 are successive sides of M_{2n}, $a_1 < b_n$. Hence $a_0 \leqq b_0 < b_1 < a_1 < b_n \leqq a_n < a_{n+1} < b_{n+1}$. Let $C_{2n} = a_0 b_1 b_2 \cdots b_n a_{n+1} \cdots a_{2n-1}$ and $D_{2n} = b_0 a_1 a_2 \cdots a_n b_{n+1} \cdots b_{2n-1}$. Then, by Lemma 1, $M + M' \geqq C + D$. But C and D are both symmetric and M and M' are equal. Therefore either C or D is a symmetric circumscribed $2n$-gon of area not greater than M_{2n}.

THEOREM 4. *If R is a convex region symmetric about a point* 0, *among the $2n$-gons of maximum area inscribed in R there is one which is symmetric about* 0.

PROOF. The proof is similar to that of Theorem 3 except that Lemma 3 is used instead of Lemma 1.

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ON r-REGULAR CONVERGENCE

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In his paper *On sequences and limiting sets* [1],[1] G. T. Whyburn introduced the notion of regular convergence. He showed that in the cases of 0 and 1 regular convergence (see definition below) that the limit of sequences of many simple topological sets is of the same type as the members of the sequence. It is the purpose of this paper to extend some of these results to higher dimensions. The lack of simple characterizations of the higher dimension sets (such as the n-sphere) makes the results much weaker than in the 0 and 1 dimensional cases.

It is assumed throughout the paper that all sets lie in a compact metric space. All our complexes and cycles will be non-oriented, and the Vietoris cycles and chains (V-cycles and V-chains) will have these as coordinates. The set of all points x whose distance from a set A is less than ϵ will be denoted by $U_\epsilon(A)$. Finally we shall denote the boundary of an r-dimensional complex (or V-chain) z^r by $\dot{z}^r$.

DEFINITION. *A sequence of closed sets (A_i) converging to a limit set A is said to converge r-regularly $(\rightarrow r)$ if for every $\epsilon>0$ there exist numbers $\delta>0$ and $N>0$ such that, if $n>N$, any r-dimensional V-cycle in A_n of diameter less than δ is ~ 0 in a subset of A_n of diameter less than ϵ. If $A_i \rightarrow sA$ for all $s \leqq r$, we write $A_i \rightarrow \leqq rA$* [1].

DEFINITION. *A Vietoris cycle $\xi^r=(x_i^r)$ is called a projection cycle if $\lim_{i\to\infty}$ (point set x_i^r) $=X$ and each $x_i^r \subset X$. Clearly X is the smallest carrier* [2] *of ξ^r.*

Note. Corresponding to any cycle $\xi^r=(x_i^r)$ of a compact set F, there always exists a projection cycle $\xi_1^r \sim \xi^r$ in F. In fact if a convergent subsequence of (x_i^r) is chosen, this set can be used as the set X of the definition.

THEOREM 1. *If $A_i \rightarrow rA$, then for any $\epsilon>0$ there exist positive numbers δ and N such that if x^{r+1} is a simplex of A_i $(i>N)$ whose boundary has a V-chain realization* [3] *of diameter less than δ, then x^{r+1} has a V-chain realization of diameter less than ϵ.*

PROOF. Let δ and N be the numbers corresponding to ϵ in the definition of r-regular convergence and consider a simplex x^{r+1} of A_i $(i>N)$

Presented to the Society, September 13, 1943; received by the editors September 30, 1943.

[1] Numbers in brackets refer to the bibliography at the end of the paper.

whose boundary has a V-chain realization $\eta^r=(y_i^r)$ of diameter less than δ. By the choice of δ and N, $\eta^r\sim 0$ in a subset of A_i of diameter less than ϵ. Thus there is a V-chain $\xi^{r+1}=(x_i^{r+1})$ of diameter less than ϵ such that $\dot{x}_i^{r+1}=y_i^r$. Clearly ξ^{r+1} is the V-chain realization of x^{r+1} of diameter less than ϵ.

THEOREM 2. *If $A_i\rightarrow\leqq rA$ then for any $\epsilon>0$ there exist positive numbers $\delta^r(\epsilon)$ and $N^r(\epsilon)$ such that if x^s $(s\leqq r+1)$ is a $\delta^r(\epsilon)$-simplex of A_i $(i\geqq N^r(\epsilon))$, then x^s has a V-chain realization in a subset of A_i of diameter less than ϵ.*

PROOF. The proof shall be by induction.

The case $r=0$ is clearly a direct consequence of the definition of 0-regular convergence where the numbers $\delta^0(\epsilon)$ and $N^0(\epsilon)$ are the numbers δ and N of the definition.

Suppose the theorem to be true for $r=k-1$ and consider the case $r=k$. Let ϵ be an arbitrary positive number and $\delta<\epsilon$, N be the numbers corresponding to it in Theorem 1. Now by hypothesis there exist positive numbers $\delta^{k-1}(\delta/3)$ and $N^{k-1}(\delta/3)$, and we define $\delta^k(\epsilon)=\min(\delta^{k-1}(\delta/3),\ \delta/3)$ and $N^k(\epsilon)=N^{k-1}(\delta/3)+N$. Let x^s $(s\leqq k+1)$ be a $\delta^k(\epsilon)$-simplex of A_i $(i>N^k(\epsilon))$. If $s\leqq k$, we know that x^s has a V-chain realization of diameter less than $\delta/3<\epsilon$ since $\delta^k(\epsilon)\leqq\delta^{k-1}(\delta/3)$ and $N^k(\epsilon)>N^{k-1}(\delta/3)$. If $s=k+1$, the boundary of x^{k+1} is a k-dimensional $\delta^{k-1}(\delta/3)$-cycle of A_i $(i>N^{k-1}(\delta/3))$. Thus each simplex of this cycle has a V-chain realization in a subset of A_i of diameter less than $\delta/3$. Furthermore we may suppose these realizations to be chosen so that common sides of two simplices have the same realizations. Now adding these V-chains for the simplices of the cycle we obtain a sequence of k-dimensional cycles, a subsequence of which yields a k-dimensional V-cycle of diameter less than $\delta^k(\epsilon)+2(\delta/3)<\delta/3+2(\delta/3)=\delta$ which is clearly a realization of the boundary of x^{k+1}. Now by the choice of δ, we know that x^{k+1} has a V-chain realization in a subset of A_i $(i>N^k(\epsilon)>N)$ of diameter less than ϵ. Thus the theorem is true for $r=k$ and hence for all cases.

THEOREM 3. *If $A_i\rightarrow\leqq(r-1)A$ and C is the smallest carrier of an essential projection cycle $\xi^r=(x_j^r)$ of A, then C can be expressed as $\lim_{i\to\infty}C_i$ where C_i is the smallest carrier of a cycle ξ_i' of A_i which will be essential for all sufficiently large i.*

PROOF. Since ξ^r is essential, there is a positive number η such that ξ^r is not $\sim_\eta 0$ in C. Let $\epsilon_k\rightarrow 0$ be a sequence of positive numbers and let $3\delta_k=\min\ \delta^{r-1}(\epsilon_k/3)$, $\delta^{r-1}(\eta/3)$ (from Theorem 2). Choose N_k' such that for $i>N_k'$ we have $U_{\delta_k}(A_i)\supset A$, $U_{\delta_k}(A)\supset A_i$ and let $N_k''=N_k'$

$+N^{r-1}(\delta_k/3)+N^{r-1}(\eta/3)$. This defines a number N_k'' for each k, and by letting $N_k=\sum_{j=1}^{k}N_j''$ we obtain a monotone increasing sequence of numbers. Consider a δ_k-cycle $x^r_{f(k)}$ of ξ^r that is not $\sim_\eta 0$ in C and such that $U_{\delta_k}(x^r_{f(k)})\supset C$. Now pick a number $n>N_k$ and let $a_0, a_1, \cdots, a_g$ be the vertices of $x^r_{f(k)}$, then for each $s\leqq g$, let b_s be a point of A_n such that $\rho(a_s, b_s)<\delta_k$. For each simplex $(a_{i_0}, a_{i_1}, \cdots, a_{i_r})$ in $x^r_{f(k)}$ let $(b_{i_0}, b_{i_1}, \cdots, b_{i_r})$ be a simplex and let x^r be the cycle consisting of these simplices. Clearly x^r will be a $3\delta_k$-cycle of A_n, and we shall call any cycle obtained in this manner a δ_k-projection of $x^r_{f(k)}$ [1]. Our choice of $3\delta_k$ allows us to realize each simplex of x^r in a subset of A_n of diameter less than $\epsilon_k/3$, $\eta/3$, from which we obtain a V-cycle realization $\xi^r_n=(x^r_{j_n})$ of x^r in $U_{\delta_k/3}(x^r)$ as well as in $U_{\eta/3}(x^r)$. Now C_n, the smallest carrier of ξ^r_n, will satisfy the conditions of our theorem, for $C_n\subset\overline{U}_{\epsilon_k/3}(x^r)$, $x^r\subset U_{\delta_k}(x^r_{f(k)})$, $x^r_{f(k)}\subset U_{\delta_k}(x^r)$, $C\subset U_{\delta_k}(x^r_{f(k)})$. Therefore, $C_n\subset U_{\epsilon_k}(C)$, $C\subset U_{\epsilon_k}(C_n)$ as $(\epsilon_k/3)+\delta_k+\delta_k<\epsilon_k/3+\epsilon_k/3+\epsilon_k/3=\epsilon_k$. Now for all n such that $N_k\leqq n<N_{k+1}$, choose C_n corresponding to ϵ_k; then $\lim_{n\to\infty}C_n=C$.

Finally, since $\xi^r_n\subset U_{\eta/3}(x^r)$, we obtain by an application of the prism construction [4] the homology $x^r_{j_n}\sim_{\eta/3}x^r$ for all j. Thus ξ^r_n is not $\sim_{\eta/3}0$ in C_n, for if it were then $x^r\sim_{\eta/3}0$; but by a δ_k-projection into C of the $\eta/3$-complex bounded by x^r we could obtain an η-complex in C bounded by $x^r_{f(k)}$, contrary to our hypothesis. This shows that ξ^r_n is essential for sufficiently large n and concludes the proof.

The necessity of the regular convergence in the preceding theorem is shown by the following example. Let A_i be the arc of the circle $\rho=1$ where θ varies from $1/i$ to $2\pi-(1/i)$; then A will be the circle $\rho=1$, and the 0-regular convergence is clearly violated. Now A is the smallest carrier of an essential 1-cycle, but the theorem cannot be satisfied, as A_i contains no essential 1-cycles.

COROLLARY 3.1. *If $A_i\to\leqq(r-1)A$, where A_i is a T_r-set* [5] *for each i, then A is a T_r-set.*

PROOF. If the theorem were not true, then T would contain an essential r-dimensional cycle; but by the theorem there would exist essential r-dimensional cycles in some of the A_i, which contradicts their property of being T_r-sets.

THEOREM 4. *If $M_i\to\leqq rM$ and B is an irreducible membrane* [2] *for the homology $\xi^r=(x^r_j)\sim 0$ in M, then B can be expressed as $\lim_{i\to\infty}B_i=B$, where $B_i\subset M_i$ is an irreducible membrane for a homology $\xi^r_i\sim 0$ in M_i.*

PROOF. Let $\epsilon_k\to 0$ be a sequence of positive numbers and let $3\delta_k=\delta^r(\epsilon_k)$ and $N^r(\epsilon_k)$ be the numbers corresponding to ϵ_k in Theorem

2. Choose N_k' such that for $i > N_k'$, $U_{\delta_k}(M_i) \supset M$, $U_{\delta_k}(M) \supset M_i$, and let $N_k'' = N^r(\epsilon_k) + N_k'$. This defines a number N_k'' for each k, and by letting $N_k = \sum_{j=1}^{k} N_j''$ we obtain a monotone increasing sequence of numbers. Since B is an irreducible membrane of $\xi^r \sim 0$ in M, we can find a V-chain $\eta^{r+1} = (y_j^{r+1})$ in M such that $y_j^{r+1} = x_j^r$ for each j, and such that the point sets y_j^{r+1} converge to B. Let $j(k)$ be chosen such that $y_{j(k)}^{r+1}$ is a δ_k-complex and $U_{\epsilon_k}(y_{j(k)}^{r+1}) \supset B$, $U_{\epsilon_k}(B) \supset y_{j(k)}^{r+1}$. Let n be any fixed number greater than N_k, and project $y_{j(k)}^{r+1}$ by means of a δ_k-projection into a $(3\delta_k)$-complex y^{r+1} of M_n. Since $3\delta_k = \delta^r(\epsilon_k)$, each simplex of y^{r+1} has a V-chain realization in a subset of M_n of diameter less than ϵ_k. Combining these realizations for all simplices of y^{r+1}, we obtain a V-chain realization $\eta_n^{r+1} = (y_{jn}^{r+1})$ of y^{r+1} in $U_{\epsilon_k}(y^{r+1})$. Now a V-cycle can be formed from a subsequence of (y_{jn}^{r+1}), and a further subsequence can be chosen so that the remaining (y_{jn}^{r+1}) converge to a set B_n'. We shall denote this subsequence by the same notation $\eta_n^{r+1} = (y_{jn}^{r+1})$ and we shall let $\xi_n^r = (y_{jn}^{r+1})$. Let $B_n \subset B_n'$ be an irreducible membrane of the homology $\xi_n^r \sim 0$ in M_n. Now $U_\epsilon(B) \supset y_{j(k)}^{r+1}$, $U_{\delta_k}(y_{j(k)}^{r+1})^r \supset y^{r+1}$, $U_{\epsilon_k}(y^{r+1}) \supset B_n' \supset B_n$, $B_n \supset y^{r+1}$ $U_{\delta_k}(y^{r+1}) \supset y_{j(k)}^{r+1}$, $U_{\delta_k}(y_{j(k)}^{r+1}) \supset B$; therefore $U_{2\epsilon_k}(B) \supset B_n$ and $U_{2\epsilon_k}(B_n)$ $\supset B$. Thus if we choose ξ_n^r and B_n in M_n corresponding to ϵ_k for all n such that $N_k \leqq n < N_{k+1}$, we shall have $\lim_{n \to \infty} B_n = B$, and the conclusion of the theorem.

The necessity of the regular convergence in the preceding theorem is shown by allowing M_i to be a totally disconnected set for each i such that $\lim_{i \to \infty} M_i = M$ is a unit interval, and hence the irreducible membrane of the homology of the 0-cycle consisting of its end points. Now clearly the convergence is not 0-regular, and the conclusion of the theorem is violated since no 0-cycle of any M_i is ~ 0.

COROLLARY 4.1. *If $M_i \to \leqq r M$ and $A_i \to A$ where A_i is an A_r-set* [5] *of M_i for each i, then A is an A_r-set of M.*

PROOF. Consider any irreducible membrane B of the homology $\xi^r \sim 0$ in M where $\xi^r \subset A$. By Theorem 4 there exist cycles ξ_i^r in M_i for each i and irreducible membranes (B_i) of the homologies $\xi_i^r \sim 0$ in M_i, such that $B_i \to B$. Also since $A_i \to A$ and $\xi^r \subset A$, we can choose a ξ_i^r in A_i for each i. By the definition of an A_r-set we have $B_i \subset A_i$ for each i; therefore $B = \lim_{i \to \infty} B_i \subset \lim_{i \to \infty} A_i = A$. Thus A is an A_r-set of M.

THEOREM 5. *If $C_i \to \leqq r C$, where C_i is the irreducible carrier of an r-dimensional projection cycle ξ_i^r for each i, then C is the irreducible carrier of a projection cycle ξ^r. Finally ξ^r will be essential if and only if all but a finite number of the ξ_i^r are essential.*

PROOF. Corresponding to $\delta_i \to 0$, we can pick a subsequence of (C_i), which we shall suppose to be the whole sequence, such that $U_{\delta_i/3}(C_i) \supset C$, $U_{\delta_i/3}(C) \supset C_i$. Let $\xi_i^r = (x_{ij}^r)$; then there exists a $\delta_i/3$-cycle $x_{in_i}^r$ such that $U_{\delta_i/3}(x_{in_i}^r) \supset C_i$, $U_{\delta_i/3}(C_i) \supset x_{in_i}^r$. Project $x_{in_i}^r$ into a δ_i-cycle x_i^r of C. Clearly $U_{\delta_i}(x_i^r) \supset C$, $U_{\delta_i}(C) \supset x_i^r$. Pick a subsequence of (x_i^r), which we shall suppose to be the whole sequence, forming a V-cycle ξ^r. Now the point sets $x_i^r \to C$ and $\xi^r = (x_i^r) \subset C$; therefore C is the irreducible carrier of ξ^r.

Now suppose that ξ^r is not essential, then $\xi^r \sim 0$ in C. Since $C_i \to \leqq rC$, we know by a theorem of G. T. Whyburn [7] that C is an lc^r; hence by a theorem of R. L. Wilder [3] the r-dimensional Betti number of C is a finite number n. A result of H. A. Arnold [8] implies that all but a finite number of the C_i have this same finite Betti number. Let $\xi_{10}^r, \cdots, \xi_{n0}^r$ be a basis for r-dimensional cycles in C, which we can choose to be projection cycles with smallest carriers $C_{01}, \cdots, C_{0n}$. By Theorem 3 there exist cycles $\xi_{1i}^r, \cdots, \xi_{ni}^r$ with smallest carriers $C_{1i}, \cdots, C_{ni}$ in C_i such that $\lim_{i \to \infty} C_{ji} = C_j$ $(j = 1, \cdots, n)$. The (ξ_{ji}^r) will be linearly independent for i greater than some integer N_1, for if not we can establish by a projection a linear dependence of the (ξ_{j0}^r). Since the Betti number of each C_i for i greater than some number N_2 is n, it follows that (ξ_{ji}^r) is a basis for cycles in C_i for $i > N = N_1 + N_2$. Thus $\xi_i^r \sim \sum_{j=1}^n \alpha_j \xi_{ji}^r$ $(i > N)$ $(\alpha_j = 0 \text{ or } 1)$, where we can suppose that N was chosen large enough so that the same linear combination holds for each i. Now by projecting the complexes bounded by $\xi_i^r + \sum_{j=1}^n \alpha_j \xi_{ji}^r$, we can establish a homology $\xi^r \sim \sum_{j=1}^n \alpha_j \xi_j^r$. But $\xi^r \sim 0$; therefore $\alpha_j = 0$ for all j. Thus $\xi_i^r \sim 0$ in C_i for $i > N$, which implies that ξ_i^r is inessential.

Conversely, if ξ^r is essential then exactly the same procedure as was used in Theorem 3 can be used to show that all but a finite number of the ξ_i^r are essential.

The necessity of the regular convergence in the preceding theorem is shown by the following example. Let C_i be the collection of points $(x = j/3^i, y = 0)$ for $j = 0, 1, \cdots, 3^i$, then $\lim_{i \to \infty} C_i = C =$ the unit interval from 0 to 1, and clearly the 0-regular convergence is violated. Now each C_i is an essential 0-dimensional V-cycle and hence is its own irreducible carrier, but C clearly cannot be the irreducible carrier of an essential 0-cycle as all 0-cycles are ~ 0 in C.

THEOREM 6. *If $B_i \to \leqq rB$, where B_i is an irreducible membrane for an homology of a projection cycle $\xi_i^r \sim 0$ in B_i for each i, then B is an irreducible membrane of an homology of a projection cycle $\xi^r \sim 0$ in B.*

PROOF. In the proof of Theorem 5 we have seen how to establish a projection cycle ξ^r in B. Furthermore by projections into B of the

chains bounded by the ξ_i^r, we can establish an homology $\xi^r \sim 0$ in B. It remains to show that ξ^r is not ~ 0 in a proper subset B' of B. To this end suppose $\xi^r \sim 0$ in B'. By Theorem 4, $B' = \lim_{i \to \infty} B_i'$ where B_i' is an irreducible membrane of the homology $\xi_i' \sim 0$ in B_i. (Since the carrier of ξ^r was chosen as the limit of the carriers of the ξ_i', we can choose the B_i' corresponding to the homologies of our original ξ_i'.) Thus $B' = \lim_{i \to \infty} B_i' = \lim_{i \to \infty} B_i = B$ and B is an irreducible membrane of $\xi^r \sim 0$ in B.

THEOREM 7. *If $M_i^r \to \leqq rM$, where M_i^r is an r-dimensional closed Cantorian manifold* [6] *for each i, then if* dim $M \leqq r$, *M is also a closed r-dimensional Cantorian manifold.*

PROOF. Since $p^r(M_i^r) \neq 0$ for each i ($p^r(M_i^r)$ = the rth dimensional Betti number of M_i^r) and the convergence is regular, it follows that $p^r(M) \neq 0$. Next suppose M' is a proper closed subset of M with $p^r(M') \neq 0$. Then there exists an essential (projection) cycle ξ^r with irreducible carrier C in M'. By Theorem 3, $C = \lim_{k \to \infty} C_k$, where C_k is the smallest carrier of an essential cycle ξ_k^r in M_k^r. Now for some integer n, C_n will be a proper closed subset of M_n^r, but by the definition of M_n^r, $p^r(C_n) = 0$ contrary to the fact that ξ_n^r is essential. Thus $p^r(M') = 0$ for every proper closed subset M' of M. Finally dim $M \geqq r$, for $p^r(M) \neq 0$, implies the existence of an essential ξ^r in M. Since by hypothesis dim $M \leqq r$, we have dim $M = r$, and M is an r-dimensional closed Cantorian manifold.

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TOPICS IN THE THEORY OF ABELIAN GROUPS. I
DIVISIBILITY OF HOMOMORPHISMS

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1. Introduction. The theory of character groups of Abelian groups has in recent years become of great importance, especially through applications to topology and algebra. The character group of G is the group H of homomorphisms of G into the real numbers mod 1. Extending this, we may consider the group $H=\text{Hom}\,(G, Z)$ of homomorphisms of G into a third group Z, or more generally, a "pairing" of groups H and G into Z: a multiplication $h\cdot g=z$, satisfying both distributive laws.

Of course the duality theorems for character groups will not hold in the more general cases; but, under certain conditions, substitutes may hold. We expect in later notes to give various facts about pairings, and the closely associated problem of divisibility by integers. In the present note, we answer the question of when a homomorphism of G into Z is divisible by an integer m; this has an immediate application to a theorem in combinatorial topology.

2. Divisibility theorems, discrete groups. We use the following notations for a group X and an integer m.

$$mX = \text{all } mx, \qquad x \in X,$$

$$_mX = \text{all } x \in X \text{ such that } mx = 0.$$

Note that $x\in mX$ means that x is divisible by m.

In the theorems below, we use

$$H = \text{Hom}\,(G, Z). \tag{1}$$

Let $H'(G')$ denote all $h(g)$, $h\in H'$, $g\in G'$.

LEMMA 1. *Suppose G' is a subgroup of G,*

$$h \in H, \qquad h' \in \text{Hom}\,(G', Z).$$

Then h' can be extended over G so that $h=mh'$ in G' if and only if:

(α) $h(G)\subset mZ$.

(β) $h'(mg)=h(g)$ *if* $mg\in G'$.

(γ) *If m' is a divisor of m, then $h'(m'g)\in m'Z$ if $m'g\in G'$.*

The necessity of the conditions is clear. We prove the sufficiency first for the case

Received by the editors November 19, 1943.

$$m = p^n, \qquad p \text{ prime.}$$

To begin with, extend h' over $mG+G'$ by setting

$$h'(mg + g') = h(g) + h'(g') \qquad (g' \in G'). \tag{2}$$

This definition is unique. For if $mg+g' = mg_1+g_1'$, then $m(g_1-g) \in G'$, and (β) gives

$$0 = h'(0) = h'[m(g_1 - g) + (g_1' - g')] = h(g_1 - g) + h'(g_1' - g'),$$

so that

$$h(g) + h'(g') = h(g_1) + h'(g_1').$$

Clearly (β) holds in $mG+G'$. To prove (γ), suppose $m = m'm''$. Take any

$$m'g = mg_1 + g' \in mG + G'.$$

Then $m'(g-m''g_1) = g' \in G'$, and hence, using (2), (α), and (γ) in G', we find

$$h'(m'g) = h(g_1) + h'[m'(g - m''g_1)] \in m'Z.$$

Let S be the set of all pairs (ϕ, G''), where

$$mG + G' \subset G'' \subset G,$$

$$\phi \in \text{Hom } (G'', Z), \qquad \phi = h' \text{ in } mG + G',$$

and such that (β) and (γ) hold with ϕ and G'' in place of h' and G'. Partially order these by setting

$$(\phi_1, G_1) \prec (\phi_2, G_2) \quad \text{if} \quad G_1 \subset G_2 \quad \text{and} \quad \phi_2 = \phi_1 \text{ in } G_1,$$

that is, ϕ_2 is an extension of ϕ_1. By[1] Zorn's lemma, there is a maximal simply ordered set S' of elements of S. Let G_0 be the set of all g in some G'' with $(\phi, G'') \in S'$, and set $\phi_0(g) = \phi(g)$ for such a (ϕ, G''). Clearly G_0 is a subgroup of G, $\phi_0(g)$ is independent of the (ϕ, G'') chosen, and $(\phi_0, G_0) \in S'$; hence it is the last element in S', and there is no later element in S. Therefore ϕ_0 cannot be further extended in G. To prove the lemma for $m = p^n$, it is sufficient to show that $G_0 = G$; for $\phi_0 = h'$ in G' by hypothesis, and (β) gives

$$m\phi_0(g) = \phi_0(mg) = h(g), \qquad m\phi_0 = h.$$

Suppose $G_0 \neq G$. Then choose an element $g_1 \notin G_0$ such that for a largest possible divisor m_1 of m, $g_1 \in m_1G$. Since $mG \subset G_0$, it follows

[1] Compare Remark 2 in §3.

that $m_1<m$, and $m_1=p^\mu$, $\mu<n$. Choose g_3 and g_2 so that

$$g_1 = m_1 g_3, \qquad g_2 = pg_1 = pm_1 g_3; \tag{3}$$

then $g_2 \in G_0$. Set $m_2=p^{n-\mu-1}$; then $pm_1m_2=m$. Since $\phi_0=h'$ in $mG+G'$, (2) gives

$$m_2\phi_0(g_2) = \phi_0(pm_1m_2g_3) = h(g_3). \tag{4}$$

Since $g_3=pm_1g_3 \in G_0$, we may choose z_2 by (γ) so that

$$\phi_0(g_2) = pm_1z_2. \tag{5}$$

Let G_1 be the subgroup of G generated by G_0 and g_1. Since p is prime, and pg_1 is in G_0 while g_1 is not, $ag_1 \in G_0$ if and only if $a \equiv 0 \pmod{p}$. Extend ϕ_0 through G_1 by setting

$$\phi_0(ag_1 + g) = am_1z_2 + \phi_0(g) \qquad (a \text{ integral}, g \in G_0). \tag{6}$$

To show that this is unique, suppose

$$ag_1 + g = bg_1 + g', \qquad g, g' \in G_0.$$

Then

$$(a-b)g_1 = g' - g \in G_0, \qquad a - b = kp,$$

and

$$\begin{aligned}[am_1z_2 + \phi_0(g)] - [bm_1z_2 + \phi_0(g')] &= kpm_1z_2 + \phi_0(g-g') \\ &= k\phi_0(g_2) - \phi_0(kpg_1) \\ &= \phi_0(kg_2 - kg_2) = 0,\end{aligned}$$

as required. Clearly ϕ_0 is now a homomorphism in G_1.

Since $mG \subset G_0$, (β) holds in G_1. To prove (γ), take any

$$m'g = ag_1 + g_0 \in G_1 \qquad (g_0 \in G_0).$$

If $m'g \in G_0$, then $\phi_0(m'g) \in m'Z$, since (γ) holds in G_0. Suppose it is not. By the choice of g_1 and m_1, $m' \leqq m_1$; hence for some m'', $m'm''=m_1$. Now by (3),

$$g_0 = m'g - am_1g_3 = m'(g - am''g_3),$$

and by (γ) in G_0, $\phi_0(g_0)=m'z_0$ for some z_0. Now by (6),

$$\phi_0(m'g) = am_1z_2 + \phi_0(g_0) = m'(am''z_2 + z_0),$$

as required.

We have now shown that $(\phi_0, G_1) \in S$. Since ϕ_0 has been extended from G_0, $(\phi_0, G_0) \prec (\phi_0, G_1)$. But this contradicts the fact that S' was maximal. It follows that $G_0=G$, and the lemma is proved for the case $m=p^n$.

For a general m, we shall use induction on the number $\rho(m)$ of distinct prime divisors of m. Say, $m=m_1m_2$, $(m_1, m_2)=1$, $\rho(m_1)$ and $\rho(m_2)$ being less than $\rho(m)$. Set

$$h_1 = m_2h', \qquad h_2 = m_1h', \quad \text{in } G'.$$

Then the conditions of the lemma hold for both h_1 and h_2, with m_1 and m_2 respectively. Property (α) for m_1 and m_2 is clear. To prove (β), we have

$$h_1(m_1g) = m_2h'(m_1g) = h'(mg) = h(g).$$

To prove (γ), suppose m_1' is a divisor of m_1 and $m_1'g \in G'$; then m_1' is a divisor of m, and (γ) for h' gives

$$h_1(m_1'g) = m_2h'(m_1'g) \in m_2(m_1'Z) \subset m_1'Z.$$

Therefore h_1 and h_2 may be extended over G, so that

$$h = m_1h_1 = m_2h_2.$$

Now choose integers r, s so that $rm_1+sm_2=1$. Then

$$rh_2 + sh_1 = (rm_1 + sm_2)h' = h' \text{ in } G',$$

$$m(rh_2 + sh_1) = rm_1m_2h_2 + sm_2m_1h_1 = h \text{ in } G,$$

so that $h'=rh_2+sh_1$ is the required homomorphism.

THEOREM 1. *For any integer m, $h \in mH$ if and only if*
(a) $h(G) \subset mZ$,
(b) $h({}_mG)=0$.

Let G' be the identity alone, and set $h'(0)=0$. We need merely prove (β) and (γ) of Lemma 1. Clearly (β) reduces to (b); (γ) is trivial.

3. Divisibility properties, topological groups. We shall consider only topological groups with the following property:

(P) For each integer $m>0$, the mapping mg of G into itself is interior, that is, maps open sets into open sets.

An equivalent statement is the following: For every neighborhood U of 0 in G and every $m>0$ there is a neighborhood U' of 0 such that $U' \subset mU$; that is, all sufficiently small elements of G are divisible by m, a solution lying in U. The property holds in the commonly used groups.

REMARK 1. It follows that mG is open in $G(m>0)$; hence, for any subgroup G', $mG+G'$ is open in G.

EXAMPLE. If G is the group of dyadic rational numbers, using the

topology of the real numbers, and $m=3$, then $3G$ contains no element of the form $1/2^n$. Thus G does not satisfy (P).

LEMMA 2. *Let G and Z be topological groups, and let* (P) *hold in G. Let h' be a homomorphism of G into Z, and set $h=mh'$. Then h' is continuous if and only if h is.*

That continuity of h follows from that of h' is simple (and does not use (P)). Suppose h is continuous. Given any neighborhood V of 0 in Z, choose a neighborhood U_1 of 0 in G so that $h(U_1)\subset V$, and choose $U\subset mU_1$. Now take any $g\in U$. Then $g=mg_1$, $g_1\in U_1$, and $h'(g)=h(g_1)\in V$, proving that h' is continuous.

By Hom (G, Z) we shall now mean the group of *continuous homomorphisms* of G into Z. We do not need to consider a topology in H here.

THEOREM 2. *If G and Z are topological groups, and* (P) *holds in G, then Lemma* 2 *and Theorem* 1 *hold.*

Using the former proofs, we need merely prove the continuity of h'. But this follows from the continuity of h and the last lemma.

REMARK 2. Since mG is open in G and h' is first extended over mG, if the proof in Lemma 1 is given in the equivalent form of extending h' over larger and larger subgroups of G (as from G_0 to G_1 in the proof given), only a finite number of steps will be required in the case of most groups G.

4. On the complete resolution of H by G. We use the following definitions and facts.[3] If H, G are paired into Z, (H, G') (the *nullifier* or *annihilator* of G' in H) is the set of all h with $h\cdot G'=0$. If $(H, G')=0$, then G' *resolves* H. (Then if $h_1\neq h_2$, there is an element $g\in G'$ with $h_1\cdot g\neq h_2\cdot g$.) If $(H, {}_mG')=mH$, G' *m-resolves* H. (Then if h is not divisible by m, there is an element $g\in {}_mG'$ such that $h\cdot g\neq 0$.) "Resolves" and "0-resolves" are equivalent. If G' m-resolves H for all integers $m\geqq 0$, G' *resolves H completely*. Z *is completely divisible* if $mZ=Z$ for each integer $m\neq 0$. The main application of the Pontrjagin duality theory to topology lies in the fact[3] that any group resolves completely and is resolved completely by its character group.

THEOREM 3. *If Z is completely divisible, then G resolves $H=$*Hom (G, Z) *completely.*

[3] Compare H. Whitney, *On matrices of integers and combinatorial topology*, Duke Math. J. vol. 3 (1937) pp. 35–45. The main statement in footnote 8 of this paper is not true.

If the groups are topological, we assume G satisfies (P).

If $h \in mH$, say $h = mh'$, then $mg = 0$ implies

$$h(g) = h'(mg) = h'(0) = 0,$$

and $h \in (H, {}_mG)$. Conversely, if $h \in (H, {}_mG)$, that is, $h({}_mG) = 0$, Theorem 1(or Theorem 2) gives $h \in mH$.

5. **Application to combinatorial topology.** Let $H^r(K, X)$ and $H_r(K, X)$ denote the rth homology and cohomology groups, respectively, of K, with the coefficient group X.

THEOREM 4. *Let Z be completely divisible. Then for any G, and $H = \mathrm{Hom}\ (G, Z)$, we have the isomorphisms*

$$H_r(K, H) \approx \mathrm{Hom}\ [H^r(K, G), Z],$$

$$H^r(K, H) \approx \mathrm{Hom}\ [H_r(K, G), Z].$$

If G and Z are topological, we assume G satisfies (P).

This is Theorem 8 of Whitney, loc. cit., with the hypothesis weakened; it is essentially the Pontrjagin Duality Theorem, slightly generalized. For the proof, we need merely apply Theorem 3 to obtain the remainder of the hypothesis.

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MODULARITY IN BIRKHOFF LATTICES

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The purpose of this note is to identify upper semi-modular lattices originally defined by G. Birkhoff[1] and subsequently studied by Dilworth[2] with those M-symmetric lattices[3] (introduced independently by the author without assumption of chain conditions) which satisfy a condition of finite dimensionality.

The definitions and notations are these. In a lattice L, $a \succ b (b \prec a)$ means that a "covers" b, that is, $a > b$, together with $a \geqq x \geqq b$ implies $x = a$ or $x = b$; $(b, c)M$ means $(a+b)c = a+bc$ for every $a \leqq c$ (where $a+b$, ab are the "join" and "meet" respectively of a, b). We say that L is *M-symmetric* if the binary relation M is symmetric; L is a *Birkhoff lattice* if

$$a, b \succ ab \text{ implies } a + b \succ a, b; \tag{1}$$

L is *of finite-dimensional type*[4] if for every $a < b$ there exists a finite "principal chain"

$$a_1 \prec a_2 \prec \cdots \prec a_n,$$

with $a_1 = a$, $a_n = b$. When a, b satisfy this condition for a specific n, we say that b *is* $n-1$ *steps over* a.

The properties of the relation M are given in part in a previous paper.[5] Additional properties needed here are contained in the following lemma.

LEMMA 1. *Suppose* $b, c \in L$. *Then*

(a) $(b, c)M$ *if and only if* $bc \leqq a \leqq c$ *implies* $(a+b)c = a$;

(b) *if* $(b, c)M$, *then* $(b', c')M$ *for* $bc \leqq b' \leqq b$, $bc \leqq c' \leqq c$.

PROOF. The forward implication in (a) is obvious. To prove the

Presented to the Society, November 27, 1943; received by the editors November 22, 1943.

[1] G. Birkhoff, *Lattice theory*, Amer. Math. Soc. Colloquium Publications vol. 25, New York, 1940, p. 62.

[2] R. P. Dilworth, *Ideals in Birkhoff lattices*, Trans. Amer. Math. Soc. vol. 49 pp. 325–353; also *The arithmetical theory of Birkhoff lattices*, Duke Math. J. vol. 8 (1941) pp. 286–299.

[3] L. R. Wilcox, *Modularity in the theory of lattices*, Ann. of Math. vol. 40 (1939) pp. 490–505; see also *A note on complementation in lattices*, Bull. Amer. Math. Soc. vol. 48 (1942) pp. 453–457.

[4] This property is weaker than finite dimensionality as used by Birkhoff (loc. cit. p. 11), even if 0 and 1 exist.

[5] L. R. Wilcox, *Modularity in the theory of lattices*, pp. 491–495.

converse, let $a \leqq c$. Then $a' = a + bc$ has the property $bc \leqq a' \leqq c$, whence

$$(a + b)c = (a + bc + b)c = (a' + b)c$$
$$= a' = a + bc.$$

To prove (b) we use the condition in (a). Let $b'c' \leqq a \leqq c'$. Then

$$(a + b')c' \leqq (a + b)cc' = ac' = a \leqq (a + b')c',$$

whence (b) follows.

THEOREM 1. *Every M-symmetric lattice is a Birkhoff lattice.*

PROOF. Suppose a, $b > ab$. Then it is immediate that $a+b > a$, b. To prove $a+b > a$, let $a \leqq c \leqq a+b$. Since $b \geqq cb \geqq ab$, we have $cb = b$ or $cb = ab$ from the hypothesis $b > ab$. If $cb = b$, then $a+b \leqq c$, whence $c = a+b$. Suppose $cb = ab$. We shall prove $(c, b)M$. Let $ab = cb \leqq x \leqq b$. Then $x = ab$ or $x = b$, whence either

$$(x + c)b = (ab + c)b = (cb + c)b = cb = x,$$

or

$$(x + c)b = (b + c)b = b = x,$$

and it follows by Lemma 1 (a) that $(c, b)M$. Now the symmetry of M yields $(b, c)M$, and thus, since $bc \leqq a \leqq c$,

$$c = (a + b)c = a.$$

In all cases $c = a+b$ or $c = a$, and consequently $a+b > a$. Similarly $a+b > b$.

REMARK. The theorem just proved generalizes the known result[6] that every modular lattice is a Birkhoff lattice, since modular lattices are M-symmetric.

In order to consider the converse of Theorem 1, let, for the purposes of the following lemmas, L be a fixed Birkhoff lattice of finite-dimensional type.

LEMMA 2. *If b, $c \in L$ and $c > bc$, then $b+c > b$.*

PROOF.[7] Observe that $b \geqq bc$; if $b = bc$, $b \leqq c$, and $b+c = c > bc = b$. If $b > bc$, then there exists $n = 1, 2, \cdots$ such that b is n steps over bc. If $n = 1$, the result is obvious from condition (1) defining a Birkhoff lattice. Suppose the result has been proved for all b, c for which b is

[6] Birkhoff, loc. cit. p. 34.

[7] This is MacLane's second "exchange axiom" in the convex lattice of all $x \geqq bc$; as such it follows for finite-dimensional lattices from remarks on p. 63 of Birkhoff. Since L need not be finite-dimensional, we give the proof in full.

k steps over bc, and let b be $k+1$ steps over bc. Clearly there exists $b' \prec b$ such that b' is k steps over bc. Since $b'c \leqq bc \leqq b'c$, we have $c \succ b'c$, and by the induction hypothesis applied to b', c it follows that $b'+c \succ b'$. But $b' \leqq (b'+c)b \leqq b$, whence $(b'+c)b = b'$ or $(b'+c)b = b$. In the latter case $b' \prec b \leqq b'+c$, and thus $b = b'+c$, whence $c \leqq b$, contrary to $c \succ bc$. Consequently $(b'+c)b = b'$. Since $b'+c,\ b \succ (b'+c)b$, (1) yields

$$b + c = b + (b' + c) \succ b.$$

LEMMA 3. *If* $b, c \in L$, $c \succ bc$, *then* $(c, b)M$, $(b, c)M$.

PROOF. If $bc \leqq a \leqq c$, then $a = bc$ or $a = c$, so that either

$$(a + b)c = (bc + b)c = bc = a,$$

or

$$(a + b)\, c = (c + b)c = c = a,$$

and $(b, c)M$. Now suppose $bc \leqq a \leqq b$. Then $bc \leqq ac \leqq bc$ yields $ac = bc$. Hence $c \succ ac$, and $a+c \succ a$ by Lemma 2. But $a \leqq (a+c)b \leqq a+c$, whence $(a+c)b = a$ or $(a+c)b = a+c$. In the latter case $a+c \leqq b$, and $c \leqq b$, which is impossible. Hence $(a+c)b = a$, and $(c, b)M$.

LEMMA 4. *Suppose* $b, c \in L$, $(b, c)M$. *Then* $bc \leqq a \leqq b$, $a+c = b+c$ *implies* $a = b$.

PROOF. If $c = bc$, that is, $c \leqq b$, or if c is one step over bc then $(c, b)M$ either by direct verification or by Lemma 3; hence

$$a = a + cb = (a + c)b = (b + c)b = b.$$

Suppose the result holds for all b, c with c n steps over bc, and let b, c satisfy the hypotheses, c being $n+1$ steps over bc. Then there exists c' with $bc \leqq c' \prec c$, where c' is n steps over bc. Since $(b, c')M$ by Lemma 1 (b), and since $bc' = bc \leqq a \leqq b$, we need only verify $a+c' = b+c'$ in order to show $a = b$. Since $(a, c)M$ by Lemma 1 (b), $(c'+a)c = c'$. Thus

$$c \succ c' = (c' + a)c,$$

and by Lemma 2,

$$c + b = c + a = c + (c' + a) \succ c' + a.$$

But

$$c' + a \leqq c' + b \leqq c + b,$$

whence

$$c' + b = c' + a \quad \text{or} \quad c' + b = c + b.$$

In the second case, since $(b, c)M$,

$$c' = (c' + b)c = (c + b)c = c,$$

which is impossible. This completes the proof.

THEOREM 2. *Every Birkhoff lattice L of finite-dimensional type is M-symmetric.*

PROOF. Suppose $(b, c)M$, and in proof of $(c, b)M$ let $bc \leqq a \leqq b$. Define

$$b_1 = (a + c)b \geqq a;$$

we shall prove that $b_1 = a$ by applying Lemma 4 to a, b_1, c in place of a, b, c. First, $(b_1, c)M$ by Lemma 1 (b), since $bc \leqq b_1 \leqq b$, and $(b, c)M$. Moreover,

$$b_1c = (a + c)cb = bc \leqq a \leqq b_1.$$

Finally, $a+c \geqq b_1, c$, whence

$$a + c \geqq b_1 + c \geqq a + c,$$

and $a+c = b_1+c$. The hypotheses of Lemma 4 have been verified, and thus $a = b_1$, as was to be proved.

The effect of Theorems 1 and 2 is to show that not necessarily finite-dimensional M-symmetric lattices are a true generalization of the Birkhoff lattices. Moreover, the condition defining M-symmetry does not lose its strength in infinite-dimensional cases as does condition (1). For example, an interval of real numbers ordered as usual satisfies (1) vacuously; it is modular, hence M-symmetric. However, define a lattice L as consisting of the closed real interval $I = [0, 1]$, ordered naturally, together with an element ϵ, with $0 < \epsilon < 1$, but $x \not< \epsilon$, $\epsilon \not< x$, $\epsilon \neq x$ for $x \in I$. This is a lattice in which the only covering relations are $\epsilon > 0$, $1 > \epsilon$. Hence (1) is vacuously true, but M-symmetry fails violently, since $(x, \epsilon)M$ for every $x \in L$, but $(\epsilon, x)M$ is false except for $x = 0$, 1 or ϵ.

Interesting questions are these. What infinite-dimensional generalization of the Jordan chain condition holds in M-symmetric lattices? Moreover, in finite-dimensional lattices, (1) together with its dual implies modularity; what can be said generally of lattices which together with their duals are M-symmetric?

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